Universidade do Estado do Rio de Janeiro
Centro de Tecnologia e Ciências
Instituto de Física Armando Dias Tavares

Ismael dos Santos Porfirio Júnior

Entanglement entropy in the quadratic sector of Gribov-Zwanziger theory

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Dissertação apresentada, como requisito parcial para obtenção do título de Mestre, ao Programa de Pós-Graduação em Física, da Universidade do Estado do Rio de Janeiro.

Orientador: Prof. Dr. Marcelo Santos Guimarães

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Banca Examinadora:

Prof. Dr. Marcelo Santos Guimarães (Orientador)
Instituto de Física Armando Dias Tavares - UERJ

Profa. Dra. Leticia Faria Domingo Palhares
Instituto de Física Armando Dias Tavares - UERJ

Prof. Dr. Antonio Duarte Pereira Junior
Universidade Federal Fluminense

Prof. Dr. Luis Esteban Oxman
Universidade Federal Fluminense

Prof. Dr. Bruno Werneck Mintz
Instituto de Física Armando Dias Tavares - UERJ

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DEDICATION

I would like to dedicate this dissertation to my wife, my mother, and my brother for their constant support, which was pivotal in this Journey. I also thank my late father and grandparents, without them none of this would have been possible.

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ABSTRACT

PORFIRIO JÚNIOR, I. S. *Entanglement entropy in the quadratic sector of Gribov-Zwanziger theory.* 2023. 66 f. Dissertação (Mestrado em Física) – Instituto de Física Armando Dias Tavares, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, 2023.

In this work, we discuss the concept of entanglement entropy and its application to realm of quantum field theory, with a focus on the Gribov-Zwanziger Theory's quadratic sector. We were able to obtain the entanglement entropy of this theory, and the results show that its confining nature significantly affects this quantity.

Keywords: Entanglement Entropy. Quantum Field Theory. Gribov-Zwanziger. Confinement.

RESUMO

PORFIRIO JÚNIOR, I. S. *Entropia de emaranhamento no setor quadrático da teoria de Gribov-Zwanziger*. 2023. 66 f. Dissertação (Mestrado em Física) – Instituto de Física Armando Dias Tavares, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, 2023.

Neste trabalho, discutimos o conceito de entropia de emaranhamento e sua aplicação ao domínio da teoria quântica de campos, com foco no setor quadrático da Teoria de Gribov-Zwanziger. Conseguimos obter a entropia de emaranhamento dessa teoria, e os resultados mostram que sua natureza confinante afeta significativamente essa quantidade.

Palavras-chave: Entropia de Emaranhamento. Teoria Quântica de Campos. Gribov-Zwanziger. Confinamento.

LIST OF FIGURES

LIST OF ABBREVIATIONS AND ACRONYMS

EE	Etanglement Entropy
QFT	Quantum Field Theory
QED	Quantum eletrodynamics
GZ	Gribov-Zwanziger
QuadGZ	Quadratic Gribov-Zwanziger

CONTENTS

INTRODUCTION

The phenomenon of entanglement is widely known to define a distinction between classical and quantum behavior. A very clear picture of this phenomenon arises if we consider a two spin system. They are said to be entangled, if some of their properties, such as spin, which is of a very fundamental nature and therefore often enough to convince one of its importance, become correlated even when they are put far apart from each other. What we mean by correlated is that both share the same wavefunction, and it cannot be broken down into the part that belongs only to system A and the one that belongs to system B exclusively. Rather, when the wavefunction states are measured by one of the systems, it alters the possible measure outcomes of the other. This intriguing behavior has driven extensive research in different areas of physics, especially because it is possible to quantify the amount of entanglement shared between systems. In order to understand that, first recall the concept of entropy.

Classical or Shannon Entropy (Shannon, 1948) is a fundamental quantity of statistical mechanics and information theory and can be associated with the degree of ignorance, in the probability distribution sense, one has about a given system. Consider we have a random variable, that is, a variable whose possible values are outcomes of a random event. These outcomes can be numerical or non-numerical, each with probability p_i of occurring, and probability distribution $P = \{p_i\}$, and obeying the normalization $\sum_i p_i = 1$. The Shannon entropy is given by

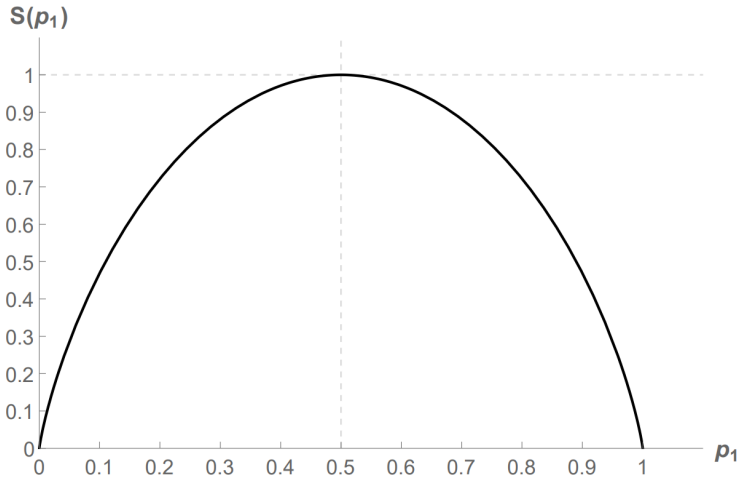
$$\mathcal{S} = -\sum_{i}^{n} p_i \log_b p_i, \tag{1}$$

where b is the logarithm basis. For instance, suppose that when we press a button, it will produce a beep or no beep. If the probability of beeping is p_1, then $p_2 = 1 - p_1$ is the probability of not beeping. Consider also the basis 2 for the logarithm since our random variable is binary. The Shannon entropy for this case will be

$$\mathcal{S} = -p_1 \log_2 p_1 - (1 - p_1) \log_2 (1 - p_1)\,. \tag{2}$$

To understand how this relates to ignorance about the outcome, let us plot the Shannon entropy as a function of p_1 Fig.(1). The graph shows that when we are certain that the beep does not occur, i.e., its probability is zero, the entropy is also zero. As the probability of beeping increases, but is still more biased toward not beeping, the entropy increases (and so our lack of certainty) until it reaches its maximum value when the probabilities are evenly split, i.e., there is no preferred result. At this point the entropy and our ignorance about the outcome is maximum. Similarly, when the outcome becomes more

Figure 1 - Shannon entropy as a function of p_1



Legend: Entropy of a binary random variable as a function of one of its probabilities p_1
Source: The author, 2023.

biased towards beeping, we begin to see a decrease in entropy until it reaches zero again when we know it will definitely produce the beep.

The following is a summary of the discussion made in Chapter 1. Entanglement entropy gives us a measure of entanglement. For some simple quantum mechanical systems, the analogy with the Shannon entropy is very clear: what we called total certainty would be no entanglement and complete ignorance is related to maximal entanglement. In quantum mechanics and quantum field theory, the term "entanglement entropy" is commonly used interchangeably with von Neumann entropy. It is usually analyzed taking into account a bipartition between two systems, obtaining their density matrices by a given method, then computing the entanglement entropy from it. This last step requires techniques associated with the particular system of study and is usually tricky to do for QFT, some noteworthy contributions to the development of EE in this context can be found in (Bombelli et al., 1986; Srednicki, 1993; Callan; Wilczek, 1994; Calabrese; Cardy, 2004). Among others (see Appendix A), the real time approach starts with the Hamiltonian of the system, and the Euclidean approach begins with a path integral formulation instead. Due to the nature of our discussion, euclidean formalism will be the one we will be emphasizing, and the calculation of its entropy utilizes the replica trick technique. Entanglement entropy is currently being applied in quantum computing, the physics of black holes, condensed matter, quantum field theory, holography, etc. It allows one to explore quantum phase transitions (Nishioka, 2018); gives us a glimpse of confinement; is related to gravity in holography (Rangamani; Takayanagi, 2017).

The concept of confinement, in general lines, mean that, even though quarks are

the constituents of hadrons, it seems to be impossible to observe them separately. This issue has been recognized in quantum chromodynamics, which is a non-abelian gauge theory as explained in Chapter 2, that deals with quark and gluon fields. In the standard model, these fields interact with each other and among themselves via the strong force as they carry color charge. Due to confinement, we cannot observe free quarks for QCD at the IR scale. The field theory of Gribov-Zwanziger (Zwanziger, 1989) provides useful tools for exploring confinement. There are other approaches to do so like lattice QCD and large N expansion. GZ considers the Gribov problem, as explained in 3. In QCD, There are many gauge fields associated with the same physical state, those are called Gribov copies and demand a gauge fixing procedure. The ambiguities are not eliminated completely at once (although it is possible in principle do so), but are restricted to a domain called Gribov region. The exact mechanisms that give rise to confinement are not understood yet, which is why it is analyzed indirectly, entanglement entropy being one of the probing tools.

Given that, we seek an analytical expression for the entanglement entropy of the quadratic sector of GZ (QuadGZ) with an interest in confinement. This is carried out in Chapter 4. It is done by analogy with a general free quantum field theory, where it has been known that the entropy is proportional to the area of the entangling surface(Srednicki, 1993). The interpretation of the result obtained is still an initial step, and we aim to better develop and generalize it in the near future.

1 ENTANGLEMENT ENTROPY

In section 1.1, the fundamentals of entanglement entropy are presented. In section 1.2, we worked out an example in quantum mechanics. We listed some crucial properties entanglement entropy enjoys in 1.3. In section 1.4 we deal with a continuous spin network and generalize to the continuum in 1.5.

1.1 Entanglement Phenomenon

An intriguing property of quantum mechanical objects is quantum entanglement. This means that some properties of one of these objects (like spin) cannot be described without referring to at least a second one. For example, let us consider a system of two Qubits (each with eigenstates up and down along the z-axis)

$$|\psi\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\uparrow\rangle_A \otimes |\uparrow\rangle_B \,, \tag{3}$$

where the total Hilbert is constructed via the tensor product of each system like $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$, and the subscripts refer to the systems A and B. Now, we can try to describe each system separately, this can be accomplished with the following factorization

$$|\psi\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle_A \otimes (|\downarrow\rangle_B + |\uparrow\rangle_B). \tag{4}$$

each state can be described on its own in the sense that we can write the total state as

$$|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B \,. \tag{5}$$

This is an example of an unentangled state. Trying to write a combined state as in Eq.(5) is a useful way to test if it is entangled or not.

In contrast, if we make a slight change and write another state of the same sort as

$$|\psi\rangle = \frac{1}{\sqrt{2}} |\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B \,, \tag{6}$$

we cannot factorize it anymore

$$|\psi\rangle \neq |\psi\rangle_A \otimes |\psi\rangle_B \,, \tag{7}$$

and Eq.(7) defines an entangled state. In order to appreciate how this is a departure

from classical physics, recall that in a phase space with positions coordinates q and momenta p, the state of the entire system (C, D) can be reconstructed from the product of states of individual subsystems C and D (Horodecki et al., 2009)

$$(C, D) = (q_C, p_C) \times (q_D, p_D), \tag{8}$$

where $\times$ denotes the Cartesian product. Whereas in Eq.(5) this is not always possible, because the state vectors can be entangled as we just discussed.

When two qubits are entangled, it means that we cannot describe each system on its own anymore, but what we do in one system will affect the other and vice-versa. For example, suppose this two-spin system is separated and sent to Lab A and Lab B. If $|\downarrow\rangle$ is measured in Lab B, that will change the total information of the combined system, thus affecting the outcome of the measure in A, leaving only $|\uparrow\rangle$ as an option. The person measuring the system in Lab A will not know if he measured $|\uparrow\rangle$ from the total combined state or because the other person in Lab B measured $|\downarrow\rangle$ first.

Although the spins seem to affect each other instantaneously, the whole process does not violate locality. In the scenario where we have two Labs, a signal would have to be sent to establish communication between them, which means that faster-than-light information transfer is not allowed. One can also wonder if there are hidden variables confusing our outcomes, this problem is what became known as the EPR (Einstein, Podolsky, and Rosen) paradox(Einstein et al., 1935). In fact, this discussion drew great attention to the phenomenon of entanglement, which ultimately raises a fundamental question about the completeness of quantum mechanics. This problem has been addressed by John Stewart Bell by showing that it is possible to statistically sort out a theory with hidden variables from a pure quantum mechanical one with a criterion known as Bell inequality(Bell, 1964). Moreover, John Clauser (Freedman; Clauser, 1972), Alain Aspect (Aspect et al., 1982) and Anton Zeilinger(Zeilinger, 1999) were able to measure that the correlations are in fact quantum mechanical, thereby winning the Nobel Prize in Physics for the year 2022. A detailed discussion about the experimental advances can be found in (Horodecki et al., 2009).

1.2 **Quantum Mechanics**

Now that we have some understanding of quantum entanglement, we can try to quantify how much the systems we are interested in are entangled. We introduce entanglement entropy Eq.(9), which is the same as the Von Neumann entropy as mentioned earlier. Henceforth we will make the supposition that we have access to subsystem A of

the bipartition $\mathcal{H}_{tot} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$.

$$S_{EE} = -Tr_A(\rho_A \ln \rho_A) \tag{9}$$

where ρ_A is the reduced density matrix

$$\rho_A = \sum_i \langle\psi|_B^i \, \rho_{tot} \, |\psi\rangle_B^i \, . \tag{10}$$

where $|\psi\rangle_B^i$ refers to each state in subsystem B. For example, in Eq.(6), $|\psi\rangle_B^1 = |\uparrow\rangle_B$ and $|\psi\rangle_B^2 = |\downarrow\rangle_B$, where $i = 1, 2$. The most general form of the total density matrix is

$$\rho_{tot} = \sum_i p_i \, |\psi\rangle \, \langle\psi| \, , \tag{11}$$

and $\sum_i |p_i| = 1$ normalize the probabilities. The reduced and total density matrices as well as the entanglement entropy are either zero or positive.

We can come back to the same systems we were using in the previous sections, namely Eq.(3) and Eq.(6), and find their reduced density matrices Eq.(10), respectively with superscripts (1) and (2)

$$\begin{aligned} \rho_A^{(1)} &= \langle\downarrow|_B \, (|\psi\rangle \, \langle\psi|) \, |\downarrow\rangle_B + \langle\uparrow|_B \, (|\psi\rangle \, \langle\psi|) \, |\uparrow\rangle_B \\ &= 1 \, |\uparrow\rangle_A \, \langle\uparrow|_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{12}$$

$$\begin{aligned} \rho_A^{(2)} &= \langle\downarrow|_B \, (|\psi\rangle \, \langle\psi|) \, |\downarrow\rangle_B + \langle\uparrow|_B \, (|\psi\rangle \, \langle\psi|) \, |\uparrow\rangle_B \\ &= \frac{1}{2} \, |\uparrow\rangle_A \, \langle\uparrow|_A + \frac{1}{2} \, |\downarrow\rangle_A \, \langle\downarrow|_A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} . \end{aligned} \tag{13}$$

And now applying Eq.(9) to each of them, we get

$$S_{EE}^{(1)} = S_A^{(1)} = -tr(\rho_A^{(1)} \ln \rho_A^{(1)}) = -Tr \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ln \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = 0 \, , \tag{14}$$

$$S_{EE}^{(2)} = S_A^{(2)} = -tr(\rho_A^{(2)} \ln \rho_A^{(2)}) = -Tr \left(\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \ln \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \right) = \ln 2. \tag{15}$$

From Eq.(14) and (15) we can see that this particular system has a lower and upper bound ranging from unentangled to maximally entangled $0 \leq S_A \leq \ln(d_A)$, where d_A is the dimensions of $\mathcal{H}_\mathcal{A}$, although these results are not readily generalized. In fact, we better understand them in analogy with the Shannon entropy (see Appendix B), the classical counterpart of von Neumann entropy, where its maximum value for a n-bit message is given by $\log n$ (Witten, 2020).

Another interesting system is the thermofield double, Eq.(16). It consist of two identical copies of a quantum system which are usually labeled "left" and "right" where $|n\rangle_L$ and $|n\rangle_R$ are the energy eigenstates (in an orthonormal basis) of the left and right systems, and $e^{-\beta E_n/2}$ is a Boltzmann factor.

$$|TFD\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\beta E_n/2} |n\rangle_L \otimes |n\rangle_R \tag{16}$$

Its reduced density matrix is obtained in the same way as the two-spin case, and it is given by

$$Tr_R \rho_{LR} = \rho_L = \frac{1}{Z} \sum_n e^{-\beta E_n} |n\rangle_L \otimes \langle n|_L. \tag{17}$$

Using the spectral decomposition of the Hamiltonian of the Left subsystem allows us to rewrite Eq.(17) as

$$\mathcal{H}_L = \sum_n E_n \, |n\rangle_L \otimes \langle n|_L \,, \tag{18}$$

$$e^{-\beta \mathcal{H}_L} = \sum_n e^{-\beta E_n} \, |n\rangle_L \otimes \langle n|_L \tag{19}$$

$$\rho_L = \frac{1}{Z} e^{-\beta \mathcal{H}_L}. \tag{20}$$

Writing the reduced density matrix in the form Eq.(20) is particularly useful because we can associate it with the known quantities of quantum statistical mechanics, which we briefly list for the sake of completeness: the partition function Z Eq.(21)) where $\mathcal{H}$ is a general Hamiltonian, the expectation value Eq.(22) for a general operator $\mathcal{O}$ and Helmholtz free energy Eq.(23).

$$Z = Tr e^{-\beta \mathcal{H}} \tag{21}$$

$$\langle \mathcal{O} \rangle = \frac{Tr(\mathcal{O}e^{-\beta\mathcal{H}})}{Z} \tag{22}$$

$$F = -\frac{1}{\beta} \ln Z. \tag{23}$$

The entanglement entropy of the thermofield double state, which we denote by $S_{TFD,L}$ Eq.(24), can be directly associated with the inverse temperature in the following manner (Nishioka, 2018)

$$\begin{aligned} S_{TFD,L} &= -Tr\left(\frac{1}{Z}e^{-\beta\mathcal{H}_L}\ln(\frac{1}{Z}e^{-\beta\mathcal{H}_L})\right) \\ &= -Tr\left(\frac{1}{Z}e^{-\beta\mathcal{H}_L}(-\beta\mathcal{H}_L + \ln Z)\right) \\ &= Tr\left(\beta\frac{1}{Z}e^{-\beta\mathcal{H}_L}\mathcal{H}_L\right) - Tr\left(\frac{e^{-\beta\mathcal{H}_L}}{Z}\ln Z\right) \\ &= \beta(\langle\mathcal{H}_L\rangle - F), \end{aligned} \tag{24}$$

from which we see that the entropy of thermofield double state reduces as the temperature becomes high.

1.3 **Properties**

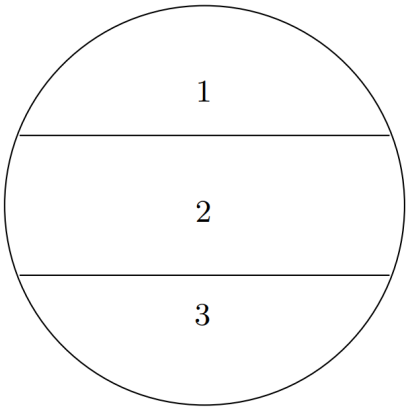
In this section, we list and briefly describe some of the properties of entanglement entropy (Von Neumann). Let us consider a bipartite system , and whose density matrix ρ_{12} describes the state of the combined system. The density matrices for the subsystems can be obtained by taking the partial trace, and we can calculate the entropies from them. If ρ_{12} is a groundstate pure density matrix, we can make the following claim about the subsystem 1 and its complement 2

$$S_1 = S_2 = 0 \tag{25}$$

This equality is easy to show from Eq.(13), that is, if we traced A instead the entropy would have been the same. Note that this will not hold anymore for finite temperature because the state of the combined system is no longer pure (Nishioka et al., 2009).

There is an important set of inequalities that relate the entropy of subsystems to

Figure 2 - Tripartion



Legend: A system divided into three disjoint subsystems 1,2 and 3
Source: The author, 2023.

the one of a combined system. The first one is called subadditivity

$$S_{12} \leq S_1 + S_2. \tag{26}$$

The subadditivity equation can be rewritten to define an important quantity known as mutual information $I(A, B)$, which is given by

$$I(1, 2) = S_1 + S_2 - S_{12} \geq 0. \tag{27}$$

It quantifies how much information is shared between two systems variables (Ince et al.,) .

Another property of any three disjoint subsystems Fig.(2) is the strong subadditivity, expressed by the inequalities

$$S_{123} + S_2 \leq S_{12} + S_{23} \tag{28}$$

$$S_1 + S_3 \leq S_{12} + S_{23}, \tag{29}$$

which applies to any three disjoint subsystems and places constraints on their entropies.

In order to illustrate these properties, let us consider the GHZ state Eq.(30) as an example. The GHZ state is a three-qubit quantum state that exhibits maximal

entanglement between its constituents.

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right), \tag{30}$$

where we now simplify the notations (for more than two spin systems, the up and down arrow notation does not facilitate the reading) using the basis vectors $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$, and $|abc\rangle = |a\rangle \otimes |b\rangle \otimes |c\rangle$. The total density matrix of the GHZ state is

$$\rho_{123} = \frac{1}{2}\left(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\right). \tag{31}$$

Computing EE for the different subsystems (Appendix C) yields

$$S_{12} = S_{23} = S_1 = S_3 = \ln 2 \tag{32}$$

$$S_{123} = 0. \tag{33}$$

where the mutual information between subsystems 1 and 2 is $\ln 2$ and it is easy to see that the strong subadditivity equations are obeyed.

1.4 **Discrete Systems**

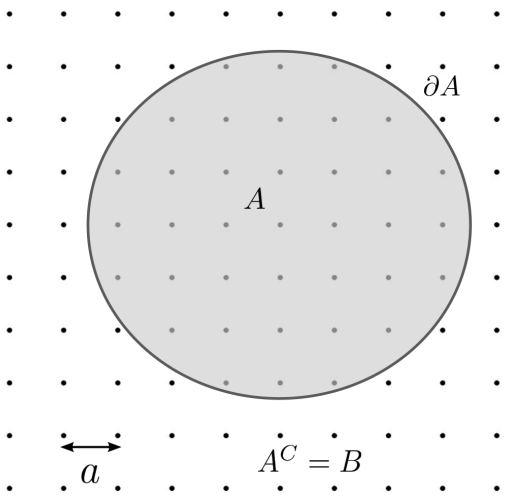
We can start by describing a lattice to help us create the intuition for the continuous systems we are actually interested in. The degrees of freedom are on finite-dimensional Hilbert Spaces on each one of the n lattice sites. The lattice space is given by a. To keep things simple, we can make an additional assumption that each site contains only 1 qubit (Rangamani; Takayanagi, 2017). Thus the total Hilbert Space $\mathcal{H}_{tot}$ may be written as

$$\mathcal{H}_{tot} = \otimes_n \mathcal{H}_n\,. \tag{34}$$

We can bipartite the whole system into regions A and B, where the boundary between them is ∂A Fig.(3). Considering the bipartition, we can reconstruct the lattice via Eq.(35).

$$\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B \tag{35}$$

Figure 3 - Discrete Bipartition



Legend: Spin Lattice bipartitioned into regions A and B
Source: The author, 2023.

The dissociation described is done thinking of an observer that has access limited to only one of these regions. We would like to tell if the degrees of freedom of such region are affected by the inaccessible one, that is if they both are entangled. If so, how much? One way to probe entanglement will be done using the density matrix Eq.(11) where the region we don't have access to ($B = A^c$) will be traced out, resulting in the reduced density matrix Eq.(10).

Finally, in order to compute the entanglement entropy, we resort to two types of measurements, the Von Neumann entropy Eq.(9) and the Rényi entropy Eq.(36), where the normalization $Tr\rho_A = 1$ was used.

$$S_A^n = \frac{1}{1-n} \ln Tr_A \rho_A^n \tag{36}$$

The relation between the two is easy[1] to show

$$\begin{aligned}\lim_{n\to 1} S_A^n &= \lim_{n\to 1} \frac{1}{1-n} \ln Tr_A \rho_A^n \\ &= -\lim_{n\to 1} (Tr_A \rho_A^n)^{-1} Tr_A \rho_A^n \ln \rho_A \\ &= -Tr_A \rho_A \ln \rho_A = S_A.\end{aligned} \tag{37}$$

In gauge theories, the physical degrees of freedom may be nonlocal in spacetime and therefore, their decomposition as in Eq.(35) is not possible. This makes it difficult to compare different regions A and B directly. To overcome this issue, a spatial lattice[2] can be used, where the physical degrees of freedom (that is, gauge invariant) are located on the links between sites, rather than the sites themselves. By doing so, a larger Hilbert space can be defined Eq.(38) that allows for a tensor product decomposition, which facilitates imensily the calculation of EE for this case (Donnelly, 2012). Although this extended definition may still be hard to tackle for to the non-abelian case, and would be probably analyzed with a more operator-oriented approach.

$$\begin{aligned}&\mathcal{H}_{\text{physical}} \subset \mathcal{H} \\ &\mathcal{H} = \mathcal{H}_{\text{physical}} \oplus \mathcal{H}^{\perp}_{\text{physical}}\end{aligned} \tag{38}$$

1.5 **Continuum**

By analogy with the previous lattice-based construction, one can define wave functionals in a QFT by choosing a suitable background spacetime to work with. To be clear, the dictionary is that the lattice is replaced by the manifold in which the QFT is defined; fields ϕ (with states $|\phi\rangle$) will substitute spins; and we will have wavefunctionals instead of wavefunctions. Consequently, we can try to write something similar to Eq.(11).

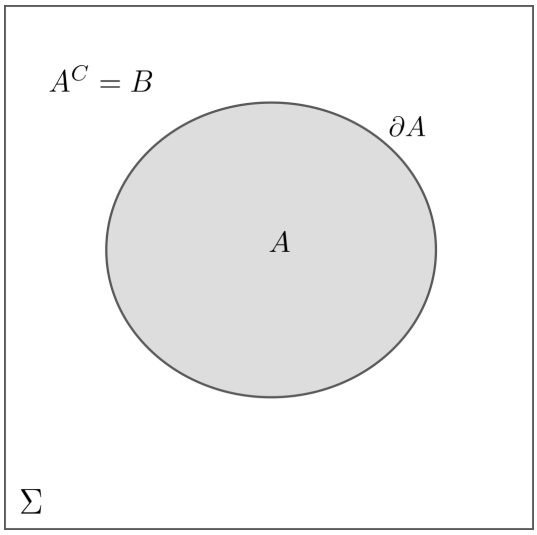
In a globally hyperbolic spacetime such as Minkowski, we are allowed to pick a Cauchy Slice Σ in it and provide a nice description of the instantaneous state of the wave functional $\Psi(\phi(x))$. The bipartition is similar: Take a co-dimension-1 region A such that $A \in \Sigma$, $B \in \Sigma$ and $A \cup B = \Sigma$ and a co-dimension-2 entangling surface ∂A separating both regions Fig.(4).

Let us first notice that the density matrix operators we will now consider will be expressed in terms of path integrals. As we are interested in entropy, we would need to calculate such an integral involving logarithms of these operators at some point, which is a

[1] We used L'Hôpital rule and the result $\frac{d}{dn} C^n = C^n \ln(C)$ in the second line.

[2] Another reason being the UV divergent behavior of EE and the lattice is a way of regulating for that.

Figure 4 - Continuous Bipartition



Legend: Continous system bipartitioned in a space-time slice by means of an entangling surface
Source: The author, 2023.

very challenging task. The trick to facilitate this calculation is to take n (integers) copies of the system using a particular path integral representation, then analytically continue it to non-integers, and finally take the limit Eq.(37) to obtain the EE. This is what became known as the replica trick (Calabrese; Cardy, 2004). We will go over this entire procedure, but first, we need to construct the wave functional to shift the discussion to QFT.

Let us consider the path integral Eq.(39), in which a Wick rotation has been performed. Here τ is the Euclidean time and S_E is the euclidean action

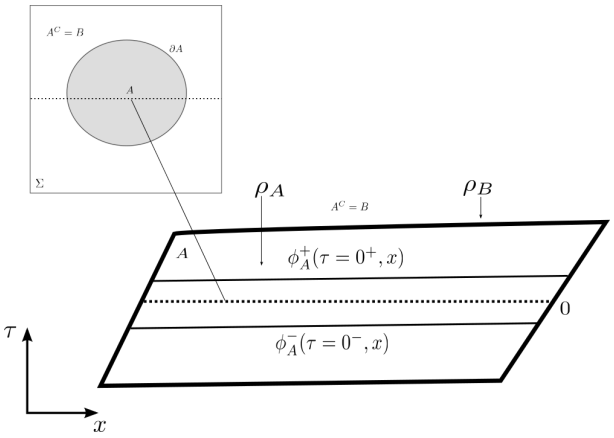
$$\langle\phi(x)|\, e^{-\hat{H}(\tau_f-\tau_i)}\, |\phi(y)\rangle = \int \mathcal{D}\phi(\tau) e^{-S_E}. \tag{39}$$

Using the completeness of energy states, $\hat{H}\,|n\rangle = E_n\,|n\rangle$, and specifying the times $\phi(\tau_i = T, y)$ and $\phi(\tau_f = 0, x)$, we can write the left-hand side as

$$\langle\phi(x)|\, e^{\hat{H}T}\, |\phi(y)\rangle = \sum_n \langle\phi(x)|n\rangle\, e^{E_n T}\, \langle n|\phi(y)\rangle = \sum_n \psi_n(x)\psi_n^*(y) e^{E_n T}. \tag{40}$$

When $T \to -\infty$ the contributions due to the groundstate dominate (Nishioka, 2018)

Figure 5 - Subregions in the Cauchy slice



Legend: A Cut along $\tau = 0$ is represented by a dotted line, and the gap between the subspaces created is exaggerated for the sake of visualization.

Source: The author, 2023.

$$\sum_n \psi_n(x)\psi_n^*(y)e^{E_n T} \approx \psi_o(x)\psi_o^*(y)e^{E_0 T}. \tag{41}$$

Using this condition in Eq.(39) and multiplying both sides of by $\int dy\psi_0(y)$, we get

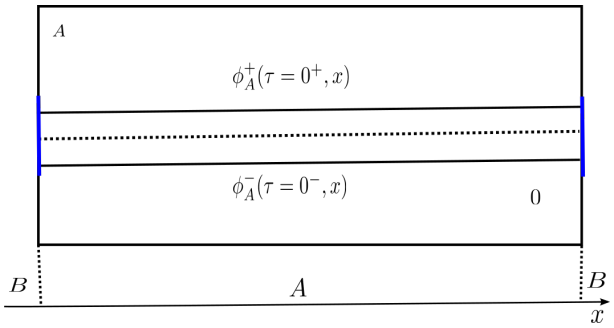
$$\Psi(x) = \int_{\tau=-\infty}^{\tau=0} \mathcal{D}\phi e^{-S_E} \tag{42}$$

where the remaining phase was absorbed, and $\Psi = \psi_0$ was changed on the left-hand side to emphasize that this is the wave functional in the ground state. Its conjugate Eq.(43) can also be obtained by similar argument

$$\Psi^*(x) = \int_{\tau=0}^{\tau=\infty} \mathcal{D}\phi e^{-S_E}. \tag{43}$$

We would like to acquire an expression for ρ_A as a path integral. For regulation purposes, we cut open around $\tau = 0$, that is 0^+ and 0^- on a Cauchy slice Fig(5), with field operators labeled by $\phi_A \to \phi_A^+; \phi_A^-$ and Φ_B. The total density matrix is simply $\rho = \frac{1}{Z} |\Psi\rangle \langle\Psi|$,the Z in the denominator is to normalize our expression . The reduced density matrix is now given by analogy with the discrete case. Taking the partial trace is now understood as integrating over the complement of A

Figure 6 - Trace of ρ_A.



Legend: The blue lines represent where the two subregions were attached.
Source: The author, 2023.

$$\rho_A = \frac{1}{Z}\int \mathcal{D}\phi_B \langle\phi_B|\Psi\rangle\langle\Psi|\phi_B\rangle. \tag{44}$$

We need to make a distinction between the fields ϕ_A^+ and ϕ_A^- acting on A $\langle\phi_A^-|\,\rho_A\,|\phi_A^+\rangle$,

$$\rho_{A-+} = \frac{1}{Z}\int \mathcal{D}\phi_B\, \langle\phi_A^-|\,\langle\phi_B|\Psi\rangle\langle\Psi|\phi_B\rangle\,|\phi_A^+\rangle \tag{45}$$

To take the trace, we can susbtitute the expressions for the wave functionals and the delta function to account for the boundary conditions we imposed on A. Fig.(6) help us visualize that, physically, the trace can be thought as the "sticking by the edges of B", that is

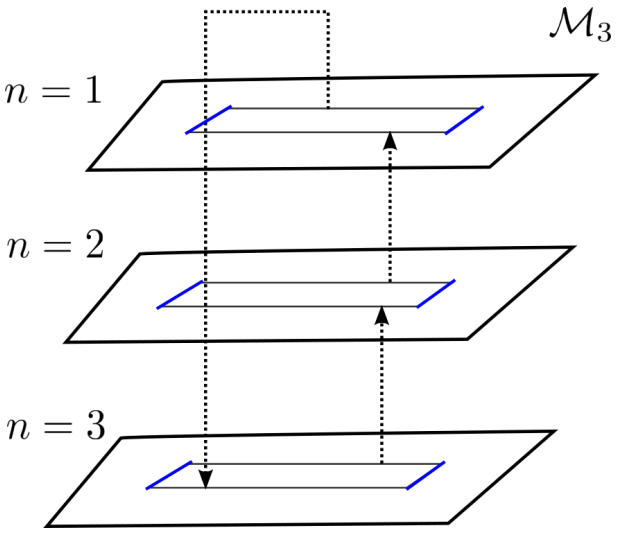
$$\rho_{A-+} = \frac{1}{Z}\int [\mathcal{D}(\tau,x)\phi] e^{-S_E} \prod_{x\in A} \delta\left(\phi(\tau = 0^-, x) - \phi_A^-\right) \delta\left(\phi(\tau = 0^+, x) - \phi_A^+\right) \ . \tag{46}$$

If we were to attach many of such path integrals together for n-sheets, we would compose a new manifold $\mathcal{M}_n$, the partition function of which will be given by

$$\ln \frac{\mathcal{Z}[\mathcal{M}_n]}{\mathcal{Z}[\mathcal{M}]^n} = Tr(\rho_A^n) \ . \tag{47}$$

This result is known and appears when studying entanglement entropy for thermal correction in two-dimensional conformal field theories (Calabrese; Cardy, 2009). Since the path integral gives us a rectangular region from the far past/future until the cut, and

Figure 7 - Path integral



Legend: Representation of the path taken on a simplified $n = 3$ manifold
Source: The author, 2023.

the trace glues such cut by the edges, taking the trace of the nth power is understood as sticking many copies successively in the new manifold, and we compute the path taken on it. Fig.(7) exemplifies this for a simple case where $n = 3$. More rigorously, what happens is that the $+$ fields living in sheet 1 are made equal to the fields to the $-$ fields living in sheet 3 and so forth, following 46.

This way we write the Rényi entropy Eq.(36) and take the $n \to 1$ limit, such that the EE is available

$$S_A = \lim_{n\to 1} S_A^n = \lim_{n\to 1} \frac{1}{1-n} \ln \frac{\mathcal{Z}[\mathcal{M}_n]}{\mathcal{Z}[\mathcal{M}]^n}. \tag{48}$$

The expected result for a free D-dimensional theory, where an expansion was done around $\epsilon = 0$ (UV-Limit) [3] and the more divergent terms were omitted is Eq.(49). We realize that the entropy is proportional to the area of the entangling surface ∂A, and the coefficients κ are proportional to the number of fields (Depending on the QFT under consideration) (Rangamani; Takayanagi, 2017). This is what became known as the area

[3] The parameter ϵ is used to regulate the region where s values are close to zero. It is squared to ensure that the exponential factor is well-behave, that is, when s is positive. The explicit use of ϵ^2 appears later in Eq.(131) when we discuss this calculation in more detail.

law.

$$S_{EE} = \kappa Area(\partial A) \left(\frac{1}{(D-2)\epsilon^{d-2}} + ... \right) \tag{49}$$

The interest in entanglement entropy has been sparked by the physics of black holes(Bombelli et al., 1986). In this context, the EE analog is the geometric or Bekenstein-Hawking entropy (Callan; Wilczek, 1994). The area law became a fundamental concept in the study of entanglement in QFT. It shows that the entanglement entropy of an arbitrary region A is proportional to the area of its boundary, rather than the volume of the region (Srednicki, 1993), which implies that entanglement entropy is a local quantity that only depends on the boundary geometry and not on the global properties of the QFT. This concept seems to be fundamental since it keeps reappearing in different contexts, for example, it is also relevant when extended to the study of holography and AdS/CFT correspondence (Ryu; Takayanagi, 2006) (Nishioka, 2018), where QFTs are related to gravity theories in higher dimensions. In the context of information theory (A recent introduction to which can be found in (Roffe, 2019)), this law is related to the concept of quantum error correction, which is a technique for protecting quantum information from the effects of noise and other forms of interference (Harlow, 2017). In our specific case, this formula will reappear in Chapter 4 when we work out an euclidean massive Free theory.

2 YANG-MILLS

One of the central pillars of contemporary particle physics is the theory of Yang-Mills. It makes sense of electroweak interaction SU(2)xU(1) and the quantum chromodynamics SU(3) of quarks and gluons. In this chapter, our main goal is to understand the symmetry principles that lead us to the construction of the pure action of Yang-Mills. We will discuss both the abelian 2.1 and non-abelian cases 2.2, which correspond to theories with and without self-interacting gauge fields, respectively.

2.1 Abelian Gauge Theory

A theory will be said to have a gauge symmetry under a certain group if the transformations are local and leave its action invariant. In this section, we will consider the abelian group U(1), that is, the one composed of phases (1×1 matrices). The transformation of a generic field φ will be

$$\varphi \to e^{iq\Gamma(x)}\varphi, \tag{50}$$

where $\Gamma(x)$ is the local symmetry parameter, and q is a constant specific to the model with which one is working. In order to be more concrete, Let us consider the Dirac action

$$S(\psi, \bar{\psi}) = \int d^4x \bar{\psi}(i\partial\!\!\!/ - m)\psi. \tag{51}$$

The fermion fields $\psi(x)$ will transform, according to Eq.(50), as

$$\psi(x) \to e^{iq\Gamma(x)}\psi(x) \quad , \quad \bar{\psi}(x) \to e^{-iq\Gamma(x)}\psi(x), \tag{52}$$

where q now is the electron charge. If we examine the action term by term, we note that the mass term is clearly invariant since the phases in Eq.(52) will simply cancel out. The kinetic term is not invariant because the transformations are local, and the derivative is taken at different points

$$\partial = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\psi\left(x + \varepsilon\right) - \psi\left(x\right) \right), \tag{53}$$

generating different symmetry parameters for each of them, that is, $\Gamma(x)$ and $\Gamma(x + \epsilon)$.

One way around this problem is to introduce a new derivative called the gauge

covariant derivative

$$\partial_\mu - iqA_\mu \equiv D_\mu. \tag{54}$$

The additional term A_μ is called a gauge field. We want Eq.(54) to transform nicely like

$$D\psi \to e^{iq\Gamma(x)} \left(\partial_\mu - iqA_\mu\right) \psi, \tag{55}$$

and this can be achieved if we let the gauge field transform in the following manner

$$A_\mu \to A_\mu + \partial_\mu \Gamma(x) = A^U_\mu. \tag{56}$$

We can check this is the case

$$\begin{aligned} D\psi \to \left(\partial_\mu - ieA^U_\mu\right) e^{iq\Gamma(x)}\psi &= e^{iq\Gamma(x)} \left(iq\partial_\mu\Gamma\left(x\right) + \partial_\mu - iqA^U_\mu\right)\psi \\ &= e^{iq\Gamma(x)}(iq\partial_\mu\Gamma\left(x\right) + \partial_\mu - iq\left(A_\mu + \partial_\mu\Gamma\left(x\right)\right)\psi \\ &= e^{iq(x)} \left(\partial_\mu - iqA_\mu\right)\psi. \end{aligned} \tag{57}$$

Eq.(51) can be promoted to the the gauge invariant Dirac action

$$S(\psi, \bar{\psi}, A) = \int d^4x \bar{\psi}(i\not{D} - m)\psi. \tag{58}$$

Finally, we are interested in the dynamics of the gauge field A_μ. It would be nice if we found a term that played this role and also was gauge invariant. This motivates us to write the commutator between the gauge covariant derivatives

$$[D_\mu, D_\nu] = -iq(\partial_\mu A_\nu - \partial_\nu A_\mu), \tag{59}$$

where we used Eq.(54), and we call the quantity

$$(\partial_\mu A_\nu - \partial_\nu A_\mu) = F_{\mu\nu} \tag{60}$$

the field-strength $F_{\mu\nu}$. It is easily shown to transform in a smooth way Eq.(61) because when it acts on ψ, it doesn't behave like a derivative, but rather like a multiplying factor, thus commuting with the exponential.

$$[D_\mu, D_\nu]\,\psi = -iqF_{\mu\nu}\psi \to iqe^{i\Gamma(x)}\left[D_\mu, D_\nu\right]\psi = iqe^{i\Gamma(x)}F_{\mu\nu}\psi \tag{61}$$

Putting all the pieces together, we arrive at the abelian gauge theory called quantum electrodynamics (QED), which describes fields (photons and electrons) and electromagnetic

forces

$$S(\psi,\bar{\psi},A)=\int d^4x\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\bar{\psi}(i\not{D}-m)\psi\right]. \tag{62}$$

The first term is clearly gauge invariant according to Eq.(61) and the factor 1/4 is a convention. We learn, therefore, that by principles of symmetry only, we can get a pretty much complete description of the fields. We move on to generalize this notion further to the non-abelian case.

2.2 Non-Abelian Gauge Theory

In order to discuss the more general non-abelian gauge symmetries, unlike in the previous section where we started with a particular action, let us make a more abstract approach and start with the transformations of terms while keeping in mind what we learned before: we have to make sure that the modifications associated with the gauge field remain gauge invariant.

Consider the non-abelian group SU(N), composed of special unitary $N\times N$ matrices, and define an infinitesimal transformation in it

$$U(x)_{jk}=\delta_{jk}-ig\theta^a(x)(T)^a_{jk}+\mathcal{O}\theta^2 \tag{63}$$

where $j,k=1,\ldots,N$ and $a=1,..,N^2-1$, g is a coupling constant, θ are real infinitesimal parameters for each of the group generators T (hermitian and traceless matrices), and we ignore terms of non-linear order(Srednicki, 2007). The commutation relation obeyed by the generators is given by

$$[T^a,T^b]=if^{abc}T^c, \tag{64}$$

here f^{abc} represents the structure constants and will specify the Lie algebra. For instance, recall that U(1) has phases as the generators so its structure constant is zero, as expected for an abelian group. On the other hand, if we pick SU(2), $a=3$ and the generators are

$$T^a=\frac{1}{2}\sigma^a \tag{65}$$

where σ^a are the pauli matrices

$$\sigma^1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\sigma^2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{66}$$

Using Eq.(64), we note that the structure constant is easily obtained as being the totally

antisymmetric Levi-Civita symbol, which appeared naturally due to the commutation relations of the Pauli matrices

$$\begin{aligned}\left[T^a, T^b\right] &= if^{abc}T^c \\ i\varepsilon^{abc}\sigma_c &= if^{abc}T^c \\ f^{abc} &= \varepsilon^{abc}.\end{aligned} \tag{67}$$

Another useful relation states that

$$Tr(T^aT^b) = \frac{1}{2}\delta^{ab}, \tag{68}$$

and we will need it for later.

Eq. (63) was presented to show the fact that θ can be expanded in terms of group generators, this notation was chosen to be suggestive of a "phase" transformation expansion in analogy with the Abelian case

$$U(x) = e^{ig\Gamma^a(x)T^a}, \tag{69}$$

here we replace θ^a with Γ^a so that the analogy becomes clearer and to point out that this factor is no longer infinitesimal (Srednicki, 2007). so a field will undergo

$$\phi \to U(x)\phi \tag{70}$$

Also by analogy with the abelian case, we can write the non-abelian covariant derivative

$$\partial_\mu - igA_\mu \equiv D_\mu. \tag{71}$$

In agreement with Eq.(70), it has to transform as

$$D_\mu \to U(x)D_\mu U^\dagger(x). \tag{72}$$

By its turn, The non-abelian gauge field A_μ transforms[4] like

$$A_\mu \to U(x)A_\mu U^\dagger(x) + U(x)\frac{i}{g}\partial_\mu U^\dagger(x) = A^U_\mu \tag{73}$$

$A'_\mu s$ will not commute Eq. (74) like in the previous section, rendering the non-abelian

[4] This is also how A_μ transforms in the abelian case, where U(x) is given by Eq.(52) and we get back Eq. (56).

field strength tensor not gauge invariant. Purely mathematically, we could explain this as being due to the matrix nature of the non-abelian gauge fields.

$$F_{\mu\nu} \equiv (\partial_\mu A_\nu - \partial_\nu A_\mu) - ig[A_\mu, A_\nu] \tag{74}$$

Once given the transformation rules of $F_{\mu\nu}$,

$$F_{\mu\nu} \to U(x) F_{\mu\nu} U^\dagger(x) \tag{75}$$

we can show that despite it not working (and by consequence also $F^{\mu\nu}F_{\mu\nu}$), coming up with a more abstract term that takes into account its trace will do the trick

$$Tr(F^{\mu\nu}F_{\mu\nu}) \to Tr(U(x)F_{\mu\nu}U(x)^\dagger U(x)F_{\mu\nu}U(x)^\dagger) = Tr(F^{\mu\nu}F_{\mu\nu}). \tag{76}$$

A theory of this type is called Yang-Mills Eq.(77) as anticipated. It gives an account of how gauge fields interact. The action resembles a lot the formulation of Maxwell. but the equations of motion that come out of it by variational principles are non-linear and difficult to solve. We will also see in the next chapter that trying to quantize it is also tricky, fortunately, there are known methods to do so.

$$S_{YM}(A) = -\frac{1}{2}\int d^4x (Tr(F_{\mu\nu}F^{\mu\nu})) \tag{77}$$

In component form, we can write the field-strength in function of the generators

$$F_{\mu\nu} = F^a_{\mu\nu} T^a \tag{78}$$

and the same is true for the gauge field

$$A_\mu = A^a_\mu T^a \tag{79}$$

substituing Eq.(78) and Eq.(79) in Eq.(74) yields

$$F^c_{\mu\nu}T^c = \partial_\mu A^c_\nu T^c - \partial_\nu A^c_\mu T^c - igA^a_\mu A^b_\nu [T^a, T^b] = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + gA^a_\mu A^b_\nu f^{abc} \tag{80}$$

That is the field-strength tensor as a function of the gauge field in component form. Lastly, we can write

$$Tr(F^{c\mu\nu}T^c F^c_{\mu\nu}T^c) = F^{c\mu\nu}F^c_{\mu\nu}(T^cT^c) = \frac{1}{2}F^{c\mu\nu}F^c_{\mu\nu} \tag{81}$$

where we used Eq.(68) in the second step. The yang-mills action is component form is

$$S_{YM}(A) = -\frac{1}{4}\int d^4x(Tr(F^{c\mu\nu}F^c_{\mu\nu})) \tag{82}$$

3 GRIBOV-ZWANZIGER

In this chapter, we are concerned about the quantization of non-abelian gauge theories. In general, the Gribov-Zwanziger action is an important instrument for comprehending the non-perturbative aspect of Yang-Mills theories. Furthermore, it facilitates significantly the analysis of confinement and the fundamental characteristics of strong interactions.

We start with the gauge fixing method in section 3.1, then discuss how it is not sufficient due to the presence of Gribov copies in section 3.2. The combined work of Gribov and Zwanziger led to the GZ action, as reported in section 3.3, which accounts for the Gribov copies, and finally, we specialize the theory for the free case in section 3.4.

3.1 Faddeev-Popov Gauge Fixing

Gauge quantum field theories, in the path integral approach, have a problem of over counting of its fields due to the gauge transformations they can be submitted to. This is a problem because it leads to divergences in the functional integral measure. A way of getting around that, at least in the perturbative regime, is the Faddeev-Popov gauge fixing procedure. Consider the path integral

$$Z = \mathcal{N} \int \mathcal{D}A e^{iS(A)} \Phi(A), \tag{83}$$

where $S(A)$ is the non-abelian gauge invariant action we described in the previous section, $\phi(A)$ are arbitrary insertions of gauge invariant operators and $\mathcal{N}$ is the normalization.

We can define a gauge-fixing function $G(A)$ made of two other terms (functions) dependent and independent of A, respectively $f(A)$ and ω

$$G(A) = f(A) - \omega. \tag{84}$$

$G(A)$ is supposed to choose only one gauge field configuration for each state. Geometrically, this is the same as saying that repeated field representations form a gauge orbit, and we want $G(A)$ to intersect each of them only once (Vandersickel; Zwanziger, 2012). The role of ω will be elucidated in a moment. The next step consists of applying a simple identity[5]

[5] Eq.(85) is a generalization of $1 = \int dx\delta x = \int du\delta u(x)\frac{du}{dx}$

$$1 = \int \mathcal{D}G\delta(G) = \int \mathcal{D}\alpha\delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right). \tag{85}$$

Here α means that quantities have been transformed; that is, they signal both gauge transformation (symmetry parameters) for the quantities in the partition function and variable change for G in Eq.(85). Moreover, that is why we need the determinant Jacobian. That being said, we can substitute and apply $S(A) = S(A^\alpha)$, $\phi(A) = \phi(A^\alpha)$ and $\mathcal{D}(A) = \mathcal{D}(A^\alpha)$ to factorize the integral in α and absorb it into the normalization

$$\begin{aligned} Z &= \mathcal{N}\int \mathcal{D}\alpha \int \mathcal{D}A\delta(f(A^\alpha) - \omega) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) e^{iS(A)}\Phi(A) \\ &= \mathcal{N}\int \mathcal{D}\alpha \int \mathcal{D}A^\alpha\delta(f(A^\alpha) - \omega) \det\left(\frac{\delta f(A^\alpha)}{\delta\alpha}\right) e^{iS(A^\alpha)}\Phi(A^\alpha) \\ &= \mathcal{N}\int \mathcal{D}\alpha \int \mathcal{D}A\delta(f(A)\omega) \det\left(\frac{\delta f(A)}{\delta\alpha}\right) e^{iS(A)}\Phi(A) \end{aligned} \tag{86}$$

The purpose of ω now becomes clearer: since Z is independent of it, we can attribute an external Gaussian integral to this variable, compensated with a factor $N(\xi)^{-1}$.The reason why we do this is to make the whole expression easier to work with, considering the familiar methods.

$$\begin{aligned} \frac{\mathcal{N}}{N(\xi)}\int \mathcal{D}\omega e^{\frac{-i\omega^2}{2\epsilon}} Z &= \frac{\mathcal{N}}{N(\xi)}\int \mathcal{D}\omega \int \mathcal{D}A\delta(f(A) - \omega) \det\left(\frac{\delta f(A)}{\delta\alpha}\right) e^{\frac{-i\omega^2}{2\epsilon}} e^{iS(A)}\Phi(A) \\ &= \mathcal{N}'\int \mathcal{D}A \det\left(\frac{\delta f(A)}{\delta\alpha}\right) e^{\frac{-if(A)^2}{2\epsilon}} e^{iS(A)}\Phi(A). \end{aligned} \tag{87}$$

It should obvious that we absorbed again the denominator $\frac{1}{N(\xi)}$ in the normalization and called it $\mathcal{N}'$. What about the determinant? By manipulating Grassmannian variables c and $\bar{c}$, we can essentially transform it into an exponential

$$\det M = \int \mathcal{D}c \int \mathcal{D}\bar{c}\, \left(e^{\int d^D x \int d^D y \bar{c}(x) M(x,y) c(y)}\right). \tag{88}$$

and Z can be rewritten as

$$Z = \mathcal{N}'\int \mathcal{D}A \int \mathcal{D}c \int \mathcal{D}\bar{c}\, \left(e^{i(S(A) - \frac{f(A)^2}{2\xi} + \int d^D x \int d^D y \bar{c}(x)\left(\frac{\delta f(A)}{\delta\alpha}\right)c(y))}\right) \tag{89}$$

where this action is now called the Fadeev-Popov action S_{FP}.

$$Z = \mathcal{N}'\int \mathcal{D}A \int \mathcal{D}c \int \mathcal{D}\bar{c}\, e^{iS_{FP}(A,c,\bar{c})} \tag{90}$$

Despite the extra fields c and $\bar{c}$ we introduced being sort of a mathematical gimmick, they

have an important function of canceling extra unphysical degrees of freedom. For this reason, they are called Fadeev-Popov ghosts, or just ghosts for short. For example, let us choose the Landau Gauge

$$G(A) = \partial_\mu A_\mu - \omega = 0. \tag{91}$$

Solving for the quadratic sector of the Fadeev-Popov action, that is

$$S_{quadFP} = A(-\partial^2)A + \bar{c}(-\partial^2)c, \tag{92}$$

we get that Eq.(90) for the case at hand is intrinsically

$$Z \approx [\big(\det(-\partial^2)\big)^{-\frac{1}{2}}]^4 \det(-\partial^2) = [\big(\det(-\partial^2)\big)^{-\frac{1}{2}}]^2. \tag{93}$$

The bosonic degrees of freedom get their contribution from $(\det(-\partial^2))^{-\frac{1}{2}}$, by its turn $\det(-\partial^2)$ is related to the fermionic ones. We knew beforehand that A has four components while only two field polarizations are allowed. The ghosts (which obey fermi statistics) cancel out exactly that leftover. The right-hand side of Eq.(93) is written cleverly to show that we end up with the correct number of two bosonic degrees of freedom, hence the squared power.

3.2 **Gribov Copies**

The relation between gauge orbits and gauge fixing was briefly mentioned in the previous section. We are now able to see how this can be understood mathematically. We have an ambiguity if we implement a gauge fixing to the field Eq.(94), but its transformation also obeys that same condition Eq.(95), where the Landau Gauge Eq.(91) was used.

$$\partial_\mu A_\mu = 0 \tag{94}$$

$$\partial_\mu A^U_\mu = 0 \tag{95}$$

The whole point of the gauge fixing was to eliminate such ambiguities, but the problem will still need to be solved entirely if this is the case. This is valid not only for Landau but for all mainly used gauge fixing schemes(Capri et al., 2014). Minor changes in Eq.(73)

lead to Eq.(96),

$$A_\mu^U = U A_\mu U^\dagger - \frac{i}{g} \left(\partial_\mu U\right) U^\dagger \tag{96}$$

before differentiating it, consider an infinitesimal gauge transform to first order $U = 1+\alpha$ and $U^\dagger = 1 - \alpha$

$$\begin{aligned} \partial_\mu A_\mu^U &= \left(\partial_\mu \alpha\right) A_\mu + A_\mu \left(\partial_\mu \alpha\right) - \frac{i}{g} \left(\partial^2 \alpha\right) \\ &= -\partial_\mu \left(\partial_\mu \alpha + ig \left[\alpha, A_\mu\right]\right) \end{aligned} \tag{97}$$

Note that the quantity inside parenthesis in the second line of equation (97)is the same as the Covariant derivative in the adjoint representation

$$\left(D_\mu \alpha\right)^a \equiv \partial_\mu \alpha^a - g f^{abc} A_\mu^b \alpha^c. \tag{98}$$

So considering back Eq.(95),we write

$$-\partial_\mu D_\mu \alpha = 0. \tag{99}$$

Accordingly, α is restricted to being a zero mode (an eigenvector with zero eigenvalues) for the operator $-\partial_\mu D_\mu$, and as a result, there is no single, unique way to determine the gauge field as pointed out initially. Consequently, there may be several gauge field configurations that describe the same physical state, which are appropriately called Gribov Copies(Gribov, 1978), the first to realize this was an issue.

Now, let's go back to the Faddeev-Popov procedure. The determinant, since omega in Eq.(91) is a constant for the functional derivative, becomes

$$\det\left(\frac{\delta f(A^U)}{\delta \alpha}\right) = \det\left(-\partial_\mu D_\mu\right). \tag{100}$$

By inserting it into the Faddev-Popov action

$$S(A) - \frac{\left(\partial_\mu A_\mu\right)^2}{2\xi} + \int d^D x \int d^D y\, \bar{c}(x) \left(-\partial_\mu D_\mu\right)(x, y) c(y), \tag{101}$$

It becomes more evident that the procedure encounters issues due to the zero modes we discussed. To work around this problem, we can consider an approximation in which the g coupling is negligible (see the second line of Eq.(98)). Naturally, this means A is small. Then, Eq.(95) reduces to

$$\partial^2 \alpha = 0. \tag{102}$$

If we compare it with the well-known eigenvalue equation

$$\partial^2\psi = \epsilon\psi, \tag{103}$$

where ψ are well behaved, we note that the zero modes (here $\epsilon = 0$) don't occur due to the ∂^2 operator being positive definite in the euclidean space

$$\left(\psi, -\partial^2\psi\right) = \epsilon > 0. \tag{104}$$

As long as the coupling is no longer ignorable, the zeros will reappear and the operator can even have negative eigenvalues. A widely discussed way to prevent this from happening is the restriction to the Gribov region Ω

$$\Omega \equiv \{A^a_\mu;\ \partial_\mu A_\mu = 0,\ \mathcal{M}^{ab}_G > 0\} \tag{105}$$

where $\mathcal{M}^{ab}_G$ is the Faddev-Popov operator inside the Gribov Region

$$\mathcal{M}^{ab}_G(x,y) = \left(-\partial^2\delta^{ab} + f^{abc}A^c_\mu\partial_\mu\right)\delta(x-y) = -\partial_\mu D^{ab}_\mu\delta(x-y)\ , \tag{106}$$

and it being positive means

$$\mathcal{M}^{ab}_G > 0 \Longrightarrow \int d^Dx \int d^Dy\ \psi^a(x)\mathcal{M}^{ab}_G(x,y)\psi^b(y) > 0 \tag{107}$$

3.3 **Gribov-Zwanziger Action**

In addition to defining the Gribov region Ω, as we did in the previous section, we need to restrict the domain of the path integral to it. The combined methods developed by Gribov and Zwanziger allow us to do this, a good discussion of which can be found in (Capri et al., 2013). Here, we are limited to presenting some key elements of the whole process. Let us start with making $Z \to Z_\Omega$, that is,

$$Z_\Omega = \int_\Omega \mathcal{D}Ae^{-S} = \int \mathcal{D}AV(\Omega)e^{-S} = \int \mathcal{D}A\mathcal{D}c\mathcal{D}\bar{c}V(\Omega)\delta\left(\partial_\mu A^a_\mu\right)e^{-S_{YM}-\int d^Dx\bar{c}(-\partial_\mu D_\mu)c}. \tag{108}$$

To perform the integration on the Gribov region, we apply the functional condition $V(\Omega)$ inside the integral Eq.(108)

Knowing that the Faddeev-Popov operator is the inverse Ghost propagator, we can

identify the zero modes of the former with poles of the latter.

$$\mathcal{G}(k,A) = \frac{\delta^{ab}}{(N^2-1)V} \langle a,k| \mathcal{M}_G^{-1} |b,k\rangle = \frac{1}{k^2}(1+\sigma(k,A)) \tag{109}$$

$\mathcal{G}(k,A)$ is the normalized trace of the Ghost propagator, which is as a function of external momenta k and the gauge fields A. $\mathcal{M}_G$ has been written in the space representation and Fourier transformed to obtain the k dependence, and σ is called the ghost form factor. This last quantity can be used to establish the proper restriction to the region, that is, make sure no poles are crossed and to what extent, thus defining a "horizon"

$$H(A) = -g^2 \int d^D p \int d^D q \left[f^{abd} \tilde{A}^d_\mu(-p) \left(\mathcal{M}_G^{-1}\right)^{bc}_{pq} f^{cae} \tilde{A}^e_\mu(q) \right], \tag{110}$$

where $H(A)$ is properly called the horizon function, and

$$\tilde{A}^c_\mu(p-q) = \int d^D x \frac{1}{(2\pi)^D} A^c_\mu(x) e^{-i(p-q)x} \tag{111}$$

are the transformations of gauge fields. If the expectation values of the Horizon function are given by

$$\langle H(A) \rangle = VD(N^2-1), \tag{112}$$

where the volume and dimension of Euclidean space-time are respectively represented by V and D, they will satisfy the no-pole condition. We can obtain the same information by considering the path integral and introducing a parameter λ

$$Z(\lambda) = e^{V\mathcal{E}(\lambda)} = \int \mathcal{D}A e^{-S} e^{\lambda\left(H(A) - VD\left(N^2-1\right)\right)} \tag{113}$$

The condition Eq.(112) is equivalent to Eq.(114) because the derivative in λ will make make it an expectation value accordingly.

$$\frac{\partial \mathcal{E}(\lambda)}{\partial \lambda} = 0 \tag{114}$$

This is called the gap equation. By finding its solution, which we will call λ^*, we are able to limit the path integral to the Gribov region and establish a definition for $V(\Omega)$ using the equation

$$Z(\lambda^*) = Z_\Omega, \tag{115}$$

where $V(\Omega)$ and Z_Ω where defined in Eq.(108). This gives rise to the Gribov-Zwanziger

non-local action

$$S_{nlGZ} = \int d^D x \left(\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + i b^a \partial_\mu A^a_\mu + \bar{c}^a \mathcal{M}^{ab}_G c^b \right) + \gamma^D H(A) - \gamma^D V D \left(N^2 - 1 \right) \tag{116}$$

From Eq.(118), we make the identification $\gamma_D = \lambda^*$, where γ has dimensions of mass, and the non locality comes from $H(A)$ Eq.(110). Finally, to make the theory local and therefore workable with QFT techniques, it is necessary to introduce auxiliary bosonic and fermionic fields ϕ and ω, respectively. Then, we have

$$\begin{aligned} \mathrm{S}_{GZ} = & \int d^D x \left(\tfrac{1}{4} F^{a\ a}_{\mu\nu\mu\nu} + i b^a \partial_\mu A^a_\mu + c^{-a} \mathcal{M}_G(A)^{ab} c^b \right) \\ & + \int d^D x \left(-\phi^{-ac}_\mu \mathcal{M}_G(A)^{ab} (\phi^{bc}_\mu + -\omega^{-ac}_\mu \mathcal{M}_G(A)^{ab} (\omega^{bc}_\mu \right) \\ & + \int d^D x \left(g \gamma^{\frac{D}{2}} f^{abc} A^a_\mu (\phi^{bc}_u + \phi^{-bc}_u) - \gamma^D D (N^2 - 1) \right) \end{aligned} \tag{117}$$

This expression is known as the Gribov-Zwanziger action Eq.(117). Let us briefly recapitulate what is going on. The first line of is the original yang-mills in the landau gauge with the implementation of the Fadeev-Popov operator $\mathcal{M}_G$, which in turn, requires the addition ghost fields $c\ c^-$. The Second line is describing auxiliary bosonic and fermionic fields ϕ and ω, respectively. The role of the auxiliary fields is to localize the theory. Finally, the third line is enforcing the restriction to the Gribov region.

3.4 **Gribov-Zwanziger Quadratic Action**

Free field theories are easier to work with than interacting theories since they typically have exact solutions and solely involve quadratic terms in the fields. The methods for quantizing fields, computing scattering amplitudes, and computing correlation functions have been known for a while and are standard textbook course material. We are interested in studying the free sector of Eq.(117), so we can utilize the usual QFT techniques. Let us define its quadratic action as (Guimarães,)

$$\mathrm{S}_{quadGZ} = \int d^D x \left(\tfrac{1}{2} A^a_\mu (-\partial^2) A^a_\mu + \bar{c}^a (-\partial^2) c^a\right)$$
$$+ \int d^D x \left(\bar{\phi}^{ac}_\mu (-\partial^2) \phi^{ac}_\mu + \bar{\omega}^{ac}_\mu (-\partial^2) \omega^{ac}_\mu + g\gamma^{\frac{D}{2}} f^{abc} A^a_\mu (\phi^{bc}_\mu + \bar{\phi}^{bc}_\mu)\right), \tag{118}$$

where the constant term has been omitted. Redefining the field

$$\phi^{ab}_\mu = \frac{1}{\sqrt{2}}(V^{ab}_\mu - iU^{ab}_\mu), \tag{119}$$

. V^{ab}_μ and U^{ab}_μ are real and the factor $\frac{1}{\sqrt{2}}$ is a convenience. By substituion of this redefinition in Eq.(118) , we obtain

$$\mathrm{S}_{quadGZ} = \int d^D x \left(\tfrac{1}{2} A^a_\mu (-\partial^2) A^a_\mu + \bar{c}^a (-\partial^2) c^a - \tfrac{1}{2} V^{ac}_\mu (-\partial^2) \phi V^{ac}_\mu\right)$$
$$+ \int d^D x \left(-\tfrac{1}{2} U^{ac}_\mu (-\partial^2) U^{ac}_\mu + \bar{\omega}^{ac}_\mu (-\partial^2) \omega^{ac}_\mu + g\gamma^{\frac{D}{2}} \sqrt{2} f^{abc} A^a_\mu V^{bc}_\mu\right). \tag{120}$$

We can simplify the expression further by diagonalization [6] of V^{ab} for $SU(N)$

$$V^{ab}_\mu = \frac{1}{\sqrt{N}} f^{abc} V^c_\mu + S^{ab}_\mu, \tag{121}$$

where

$$V^a_\mu = \frac{1}{\sqrt{N}} f^{abc} V^{bc}_\mu, \tag{122}$$

and

$$f^{abc} S^{ab}_\mu = 0. \tag{123}$$

[6] Note that $f^{abc} f^{abd} = N\delta^{cd}$.

The resulting expression is now

$$\begin{aligned}\mathrm{S}_{quadGZ} = &\int d^Dx \left(\tfrac{1}{2}A^a_\mu(-\partial^2)A^a_\mu + \bar{c}^a(-\partial^2)c^a - \tfrac{1}{2}V^a_\mu(-\partial^2)\phi V^a_\mu\right) \\ &+\int d^Dx \left(-\tfrac{1}{2}S^{ac}_\mu(-\partial^2)S^{ac}_\mu - \tfrac{1}{2}U^{ac}_\mu(-\partial^2)U^{ac}_\mu + \bar{\omega}^{ac}_\mu(-\partial^2)\omega^{ac}_\mu + g\gamma^{\frac{D}{2}}\sqrt{2}f^{abc}A^a_\mu V^{bc}_\mu\right).\end{aligned} \tag{124}$$

Finally, we can add the complex fields

$$\eta^a_\mu = A^a_\mu + iV^a_\mu \qquad \bar{\eta}^a_\mu = A^a_\mu - iV^a_\mu, \tag{125}$$

which lead to

$$\begin{aligned}\mathrm{S}_{quadGZ} = &\int d^Dx \left(\tfrac{1}{4}\eta^a_\mu(-\partial^2 - ig\sqrt{2N}\gamma^{\frac{D}{2}})\eta^a_\mu + \tfrac{1}{4}\bar{\eta}^a_\mu(-\partial^2 + ig\sqrt{2N}\gamma^{\frac{D}{2}})\bar{\eta}^a_\mu\right) \\ &+\int d^Dx \left(-\tfrac{1}{2}S^{ac}_\mu(-\partial^2)S^{ac}_\mu - \tfrac{1}{2}U^{ac}_\mu(-\partial^2)U^{ac}_\mu + \bar{\omega}^{ac}_\mu(-\partial^2)\omega^{ac}_\mu + \bar{c}^a(-\partial^2)c^a\right).\end{aligned} \tag{126}$$

The free theory as expressed in Eq.(126) is what we wanted because it can produce excitations that correspond to particles with non-physical complex masses (i-particles) $m = \pm ig\sqrt{2N}\gamma^{\frac{D}{2}}$ (Baulieu et al., 2010). We can talk about confinement in the sense that the complex poles will not allow them to be interpreted as usual particles.

If γ, the parameter in the theory, were to be equal to zero, then the condensate would also be null. This would bring us back to the original perturbative vacuum of Yang-Mills. This is because in such a case, the auxiliary fields in the theory would cancel each other out in the path integral. In other words, the physical degrees of freedom would once again be only the transverse polarizations of the gauge field. However, if $\gamma \neq 0$, such polarizations would mix with the modes of the auxiliary fields and there would be no cancellation, which indicates that the excitations can be interpreted as particles. Another point for the attentive reader is the presence of a mass dimension two condensate, which makes one wonder whether the vacuum of the theory could favor other condensates of the same sort. The answer is that this is indeed the case, and it leads to a modified version of the theory called the Refined-Gribov-Zwanziger theory.

In our specific case, the blunt way to think about it is that the partition function needed for the study of EE Eq.(47) is easier to obtain for this version of the theory (Given known techniques and given the previously mentioned efforts in that regard), and we will carry this out in the next section. Therefore, we have a possible recipe for probing the effects of confinement on quantum entanglement entropy for this particular case and would like to see if the result is something interesting.

4 ENTANGLEMENT ENTROPY IN QUADGZ

Our objective is to derive the equation (48) for the quadratic GZ and solve it. In section 4.1, we will elaborate the procedure for calculating EE for an Euclidean free theory, based mainly on the work of (Nishioka, 2018). We will then extend this method to the quadGz case in section 4.2.

4.1 Euclidean Free Theory

The prescription of interest for calculating entanglement entropy in QFT, as discussed in detail in section 1.5, is to take the limit

$$S_{EE} = -\partial_n \ln Z[\mathcal{M}_n]|_{n=1} + \ln Z[\mathcal{M}]. \tag{127}$$

While the replica trick is a formidable tool, we realize that it can be challenging to explicitly calculate entropy in general on the n-folded manifold. Fortunately, for a euclidean free theory Eq.(128), we can make a geometric correspondence between the n-fold cover and a flat cone, in such a way that it makes the calculation tangible(Callan; Wilczek, 1994).

$$\mathcal{Z} = \int \mathcal{D}\phi e^{\int d^D \phi(\nabla^2 - m^2)\phi} \tag{128}$$

In order to do that, consider the euclidean space-time coordinates (x_i, τ). We define the half-space A, with coordinates $x_1 > 0$ and B $x_1 \leq 0$. The entangling surface ∂A is given by a $D-2$ transverse space through $x_1 = 0$. The entangling surface becomes the singularity at the tip of the cone Fig.(8). The cone coordinates are (r,θ) , where $r \geq 0$ and $0 \leq \theta \leq 2\pi n$. This allows us to decompose the manifold into

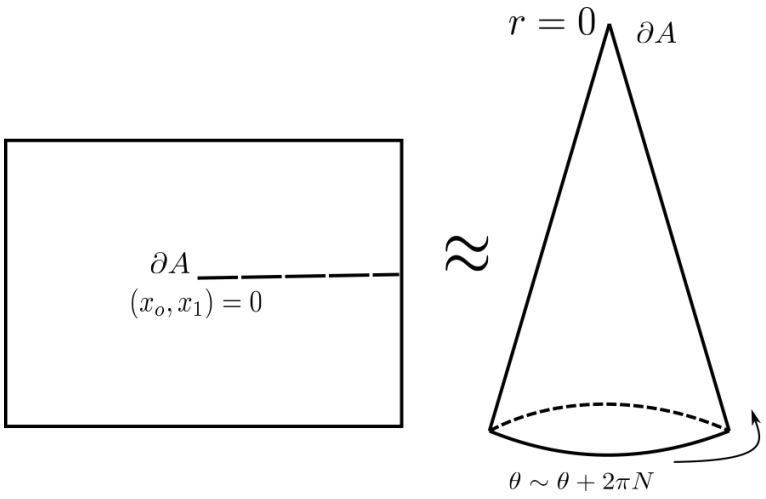
$$\mathcal{M}_n = \mathcal{C}_n \times \mathbb{R}_{D-2}. \tag{129}$$

The partition function on the $\mathcal{M}_n$ will be given by[7] the one-loop determinant

$$\ln \mathcal{Z}[\mathcal{M}_n] = -\frac{1}{2} \ln \, det(-\nabla^2 + m^2). \tag{130}$$

[7] A recent discussion can be found at (Law; Parmentier, 2022)

Figure 8 - Cone



Legend: The half space (left) is identified with a cone with a singularity at its tip (right).
Source: The author, 2023.

We can write the heat kernel to help us compute the determinant in Eq.(130)

$$\ln \mathcal{Z}[\mathcal{M}_n] = \frac{1}{2}\int_{\epsilon^2}^{\infty} \frac{\mathrm{d}u}{u} \operatorname{tr}\left[e^{-u\left(-\nabla^2+m^2\right)}\right]. \tag{131}$$

Because of Eq.(129) structure, the laplacian operator will act separately on the cone and transverse space eigenfunctions. In an orthonomal basis, we can write the eigenvalue equation as (Kabat, 1995)

$$-\nabla^2 \phi_{k,l}\phi_{\boldsymbol{k}_\perp} = (k^2 + \boldsymbol{k}_\perp^2)\phi_k\phi_\perp. \tag{132}$$

Here the eigenfunction is a product between the cone wave functional in polar coordinates, where n is a positive integer and $J_{|l/n|}(kr)$ is the Bessel function of the first kind of order $|l/n|$, and the transverse space (plane wave) $\phi_{\boldsymbol{k}_\perp}(y) = \frac{e^{(\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{y})}}{(2\pi)^{(d-2)/2}}$

$$\phi_{k,l}(r,\theta)\phi_{\boldsymbol{k}_\perp}(y) = \sqrt{\frac{k}{2\pi n}} e^{\mathrm{i}l\theta/n} J_{|l/n|}(kr) \frac{e^{(\mathrm{i}\boldsymbol{k}_\perp\cdot\boldsymbol{y})}}{(2\pi)^{(d-2)/2}}, \tag{133}$$

$$l \in \mathbb{Z}, \quad k \in \mathbb{R}^+, \quad \mathbf{k}_\perp \in \mathbb{R}^{d-2}$$

,

$$\int d^D x \phi_{k,l}\phi_{\boldsymbol{k}_\perp}\phi^*_{k',l'}(k',l')\phi_{k'_\perp} = \delta_{l,l'}\delta(k-k')\delta^{(d-2)}(k_\perp - k'_\perp). \tag{134}$$

Having the explicit form of the eigenfunction, we can calculate the trace in Eq.(131)

$$Tr[e^{-(-\nabla^2+m^2)}] = \int_{\mathcal{C}_n} \mathrm{d}^2x \sum_{l=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}k e^{-u\left(k^2+m^2\right)} \phi_{k,l}(x)\phi^*_{k,l}(x)$$
$$\int_{\mathbb{R}^{D-2}} \mathrm{d}^{D-2}y \int \mathrm{d}^{D-2}\boldsymbol{k}_\perp e^{-u\boldsymbol{k}_\perp^2} \phi_{\boldsymbol{k}^\perp}(y)\phi^*_{\boldsymbol{k}^\perp}(y) \tag{135}$$

Where the integral in y just give as the "volume" (actually the perimeter)

$$\int_{\mathbb{R}^{D-2}} \mathrm{d}^{D-2}y = \mathrm{Vol}(\mathbb{R}^{D-2}). \tag{136}$$

The integral in k perperdincular is gaussian and yields

$$\int \mathrm{d}^{D-2}\boldsymbol{k}_\perp e^{-u\boldsymbol{k}_\perp^2} = \sqrt{\frac{\pi^{D-2}}{u^{D-2}}}. \tag{137}$$

By substituting these last results back into Eq.(135), we obtain

$$Tr[e^{-(-\nabla^2+m^2)}] = \mathrm{Vol}(\mathbb{R}^{D-2})\sqrt{\frac{\pi^{D-2}}{u^{D-2}}} e^{-um^2} \frac{1}{2\pi^{D-2}} \int_{\mathcal{C}_n} \mathrm{d}^2x \sum_{l=-\infty}^{\infty} \int_0^{\infty} \mathrm{d}k e^{-uk^2} \frac{kJ^2(kr)}{2\pi n} \tag{138}$$

Making use of the following identities for the Bessel function and its modified version , that is

$$\int_0^{\infty} \mathrm{d}k\ k e^{-uk^2} J^2(kr) = e^{-r^2/2u} \frac{1}{2u} I\left(\frac{r^2}{2u}\right) \tag{139}$$

$$\int \mathrm{d}r\ r e^{-r^2/2u} I\left(\frac{r^2}{2u}\right) = -\frac{2ul}{2n} \tag{140}$$

I being the modified Bessel function of the first kind. Now, writing the remaining integral meausure in polar coordinates and using the second identity

$$Tr[e^{-(-\nabla^2+m^2)}] =$$
$$\frac{1}{2u}\frac{1}{2\pi n} \mathrm{Vol}(\mathbb{R}^{D-2})\sqrt{\frac{\pi^{D-2}}{u^{D-2}}} e^{-um^2} \frac{1}{2\pi^{D-2}} \sum_{l=-\infty}^{\infty} \int_0^{2\pi n} \mathrm{d}\theta \int \mathrm{d}r\ r e^{-r^2/2u} I\left(\frac{r^2}{2u}\right) \tag{141}$$
$$= \frac{1}{2\pi n} \mathrm{Vol}(\mathbb{R}^{D-2})\sqrt{\frac{\pi^{D-2}}{u^{D-2}}} e^{-um^2} \frac{1}{2\pi^{D-2}} \sum_{l=-\infty}^{\infty} \int_0^{2\pi n} \mathrm{d}\theta \left(-\frac{l}{2n}\right).$$

Integrating in θ and regularizing[8] the sum in l , that is, $\sum_{l=-\infty}^{\infty}|l| = -\frac{1}{6}$

$$Tr[e^{-(-\nabla^2+m^2)}] = \frac{1}{2\pi n}\,\mathrm{Vol}(\mathbb{R}^{D-2})\sqrt{\frac{\pi^{D-2}}{u^{D-2}}}e^{-um^2}\frac{1}{2\pi^{D-2}}\sum_{l=-\infty}^{\infty}2\pi n\left(-\frac{l}{2n}\right). \tag{142}$$

Finally, by grouping common factors together we get that

$$Tr[e^{-(-\nabla^2+m^2)}] = \frac{1}{12n(4\pi u)^{(D-2)/2}}\,\mathrm{Vol}(\mathbb{R}^{D-2})e^{-um^2} \tag{143}$$

Now that we have the result for the trace Eq.(143), let us go back to general formula of the EE in Eq.(127) and substitute

$$\begin{aligned} S_{EE} &= \frac{1}{2}\,\mathrm{Vol}(\mathbb{R}^{D-2})\int_{\epsilon^2}^{\infty}\frac{du}{u}\;e^{-um^2}\left[\left(\frac{1}{12n^2(4\pi u)^{(D-2)/2}}\right)\Big|_{n=1}+\left(\frac{1}{12(4\pi u)^{(D-2)/2}}\right)\right] \\ &= \frac{\pi}{3}\,\mathrm{Vol}(\mathbb{R}^{D-2})\int_{\epsilon^2}^{\infty}du\;e^{-um^2}\left(\frac{1}{(4\pi u)^{D/2}}\right). \end{aligned} \tag{144}$$

Note that the expression is convergent if $D > 2$. To solve the previous expression, we make a change of variable $t = um^2$ and identify this form as the incomplete gamma function Eq.(146) $\gamma(s,x)$, which can expanded as the power series given in Eq.(147).

$$S_{EE} = \frac{\pi}{3}\,\mathrm{Vol}(\mathbb{R}^{D-2})\int_{\frac{\epsilon^2}{m^2}}^{\infty}e^{-t}t^{D/2-1}\frac{D}{2}\left(\frac{4\pi}{m^2}\right)^{-\frac{D}{2}}dt \tag{145}$$

$$\gamma(s,x) = \int_x^{\infty}dt\left(t^{s-1}e^{-t}\right) \tag{146}$$

$$\gamma(s,x) = x^s e^{-x}\sum_{n=0}^{\infty}\frac{(s)_n x^n}{n!} \tag{147}$$

In particular, $s = D/2$ and $x = \epsilon^2/m^2$ and $(s)_n = s(s+1)(s+2)\cdots(s+n-1)$ is the Pochhammer symbol. Substituing the expansion back in Eq.(146), we have the following

[8] By using a zeta function regularization one can assign a finite value to the divergent sum over angular momentum states $\sum_{l=-\infty}^{\infty}|l| = 2\zeta(-1) = -\frac{1}{6}$ (Nishioka, 2018).

expression

$$S_{EE} = \frac{\pi}{3}\,\mathrm{Vol}\left(\mathbb{R}^{D-2}\right)\left(\frac{m}{2}\right)^{D}\left[\frac{\epsilon^{D}}{m^{D}}e^{-\epsilon^{-2}/m^{2}}\sum_{n=0}^{\infty}\frac{(D/2)_n}{n!}\left(\frac{\epsilon}{m^{2}}\right)^{n}\right] \tag{148}$$

Finally, we can further expand the exponential term to the first orders (see Appendix D) and simplify constants to obtain (Nishioka, 2018)

$$S_{EE} = \frac{\mathrm{Vol}\left(\mathbb{R}^{D-2}\right)}{6(4\pi)^{D-1}}\left(\frac{1}{(D-2)\epsilon^{D-2}} - \frac{m^{2}}{(D-4)\epsilon^{D-4}} + \frac{m^{4}}{8(D-6)\epsilon^{D-6}} + \cdots\right) \tag{149}$$

The area law appears as expected, and in addition, we have the mass contribution even in the first corrections. The presence of the mass correction is significant because it indicates that the entanglement entropy is sensitive to the properties of the underlying theory, beyond just the geometry of the entangling region. This is not contradictory to the discussion we had about area law in Section 1.5, as the area contribution still trumps the subleading ones.

In particular, the mass correction implies that the entanglement entropy can serve as a probe of the mass spectrum in a given theory. This formula also provides a paradigm to compare different types of theories, namely massive, massless, and confining.

4.2 **QuadGz**

Finally, we can adjust what we just learned from the previous section to the particular case of Eq.(126) , we have the following partition function

$$Z_{quadGZ} = \int \mathcal{D}\eta\mathcal{D}\bar{\eta}\mathcal{D}c\mathcal{D}\bar{c}\mathcal{D}S\mathcal{D}U\mathcal{D}\omega\mathcal{D}\bar{\omega}e^{-iS_{quadGz}}, \tag{150}$$

where we simplified the notation again $\mathcal{Z}[\mathcal{M}_n]_{quadGZ} = Z_{quadGZ}$. We proceed by utilizing Eq.(129) to add the quadratic operators $Q_i = -\nabla_i^2 + m_i^2$ for each field separately

$$\ln Z_{quadGZ} = -\sum_i \frac{1}{2} log\, det(Q_i). \tag{151}$$

In general, the signs may change for fermionic and bosonic fields, but that doesn't matter here since we will concentrate on the confining sector, for reasons explained earlier. Thus, we can define the confining action

$$\mathrm{S}_c \equiv \int d^{D}x\left(\tfrac{1}{4}\eta_\mu^a(-\partial^2 - ig\sqrt{2N}\gamma^{\frac{D}{2}})\eta_\mu^a + \tfrac{1}{4}\bar{\eta}_\mu^a(-\partial^2 + ig\sqrt{2N}\gamma^{\frac{D}{2}})\bar{\eta}_\mu^a\right), \tag{152}$$

Following the same rationale, we introduce the confining partition function

$$\ln Z_c = -\frac{1}{2}\left(\ln\det(\partial^2 - ig\sqrt{2N}\gamma^{\frac{D}{2}}) + \ln\det(\partial^2 + ig\sqrt{2N}\gamma^{\frac{D}{2}})\right). \tag{153}$$

Now we can use Eq.(149) to evaluate the contributions of each field to the entropy, where $m^2 = -ig\sqrt{2N}\gamma^{D/2}$ and $\bar{m}^2 = ig\sqrt{2N}\gamma^{D/2}$ are the complex masses

$$\begin{aligned}
&S_{EE} + \bar{S}_{EE} = \\
&\frac{\pi\,\mathrm{Vol}\left(\mathbb{R}^{D-2}\right)}{6(4\pi)^{D-1}}\left(\frac{1}{\epsilon^{\mathrm{D}-2}(D-2)} - \frac{m^2}{\epsilon^{\mathrm{D}-4}(D-4)} + \frac{m^4}{8\left(\epsilon^{\mathrm{D}-6}(D-6)\right)}\right) \\
&+ \frac{\pi\,\mathrm{Vol}\left(\mathbb{R}^{D-2}\right)}{6(4\pi)^{D-1}}\left(\frac{1}{\epsilon^{\mathrm{D}-2}(D-2)} - \frac{\bar{m}^2}{\epsilon^{\mathrm{D}-4}(D-4)} + \frac{\bar{m}^4}{8\left(\epsilon^{\mathrm{D}-6}(D-6)\right)}\right) \\
&= \frac{\pi\,\mathrm{Vol}\left(\mathbb{R}^{D-2}\right)}{6(4\pi)^{D-1}}\left(\frac{2}{\epsilon^{\mathrm{D}-2}(D-2)} - \frac{m^2+\bar{m}^2}{\epsilon^{\mathrm{D}-4}(D-4)} + \frac{m^4+\bar{m}^4}{8\left(\epsilon^{\mathrm{D}-6}(D-6)\right)}\right) \\
&= \frac{\pi\,\mathrm{Vol}\left(\mathbb{R}^{D-2}\right)}{6(4\pi)^{D-1}}\left(\frac{2}{\epsilon^{\mathrm{D}-2}(D-2)} - \frac{\mathrm{N}g^2\gamma^D}{2\left(\epsilon^{\mathrm{D}-6}(D-6)\right)}\right)
\end{aligned} \tag{154}$$

Duue to the nature of the expansion, the complex masses cancel each other out in the first subleading term, resulting in an entropy similar to that of a massless theory. However, it is important to note that the cancellation of the complex mass terms is a feature arising from the perturbative expansion and may not hold in all cases. In particular, non-perturbative effects, such as those arising from topological defects (Roy; Saleur, 2022; Kitaev; Preskill, 2006) can lead to additional contributions to the entanglement entropy that do not depend solely on the masses of the particles.

The Gribov Parameter, however, is already present in the next subleading term, which is indicative that the confining aspect of the theory contributes to the entropy of entanglement. In order not to loose sight of what we are discussing and to appreciate the result, let us take a step back and recall that when considering the entropy of entanglement between two subsystems of a massive scalar field, we are interested in the correlations between the degrees of freedom of the region to which we have and do not have access to, sometimes also called the "inside" and "outside" regions. For a free massive scalar theory in general, such degrees of freedom are given by the field configurations. Thus, the mass of the theory directly influences the way such field configurations entangle with one another, besides the always present and dominant boundary area contribution, which would be present even for a massless theory. For the specific case of the GZ, there is a non-trivial contribution to the degrees of freedom, in this case, the configurations of the gauge field, due to the restriction to the Gribov region. That is, the usual mass contribution vanished (and this has to do with it being non-physical), but it carried the dependence on the

Gribov parameter, and because of that, such masses left a signature on the entanglement entropy in the next order of expansion.

CONCLUSION

We started from the basics and introduced the concept of quantum entropy, stressing some key aspects of it, and leading up to its application to the realm of QFT. To clarify this application, we drew parallels between this area of research and the one in classical and information theory. This is important to make clear that even though this area of research is new, the community has been finding applications to different problems in various contexts. Next, we discussed gauge symmetries and their subtleties in some detail, such as gauge fixing and the problem of Gribov copies. This led us to the GZ theory, which is a rich environment for the study of confinement. Then, we described the square sector, which contains the confined masses that interest us.

We obtained a general expression for EE with a UV regulator from the Euclidean free theory and used the same principle to investigate the quadratic sector of the Gribov-Zwanziger theory. With that, we observed and interpreted that the restriction to the Gribov region affects the entanglement between the field configurations of the two regions through the terms in the expansion of Eq.(149), which is not a small or trivial contribution.

There are still some open questions related to this work. In particular, it is important to investigate whether mutual information cancels out divergences in a consistent and meaningful way for this specific scenario, as this could provide a useful tool for extracting more relevant information from entanglement entropy calculations in field theories. It is worth noting that we naively treated γ as a free parameter when discussing its relation to entropy, but upon considering the gap equation discussion in Section 3.3, the construction on the $\mathcal{M}_n$ manifold probably influences $\mathcal{E}(\lambda)$, so it is important to check if it is actually the case and what it entails. We also left the result in general D-dimensions, but from Eq.(154) we see that the desired entropy contribution is noticed for D=6 due to epsilon dependence. Extending this type of calculation to more general geometries or finding different expressions for entanglement entropy that allow for more insights remains a significant challenge. On the whole, this work highlights the importance of entanglement entropy calculations in field theories and the potential insights they can provide into complex physical phenomena.

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APPENDIX A – Other Methods

A.1 Mass Expansion Method

In order to calculate EE for a free scalar theory at its ground state, the following prescription will be presented. To begin with a simpler example, consider a system of only two coupled harmonic oscillators (Srednicki, 1993). Its hamiltonian is given by Eq(155). Some of them lie in an inside region and some of them in an outside one, say subscript 1 is for the former and 2 is for the latter.

$$H = \frac{1}{2}[p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2] \tag{155}$$

The ground state wavefunction is given by Eq.(156)

$$\psi_0(x_1, x_2) = \pi^{-\frac{1}{2}}(\omega_+\omega_-)e^{-\left(\frac{\omega_+ x_+^2 + \omega_- x_-^2}{2}\right)} \tag{156}$$

$$\text{where } x_\pm = \frac{(x_1 \pm x_2)}{\sqrt{2}},\ \omega_+ = k_0^{1/2},\ \omega_- = (k_0 + 2k_1)^{1/2}$$

Suppose we want to obtain the reduced density matrix for the outside region, then we can trace out the inside region by computing the gaussian integral Eq.(157), where $\beta = \frac{1}{4}\frac{(\omega_+ - \omega_-)^2}{(\omega_+ - \omega_-)}$ and $\gamma - \beta = \frac{2(\omega_+ - \omega_-)}{(\omega_+ - \omega_-)}$.

$$\rho_{out} = \int_{-\infty}^{\infty} dx_1 \psi_0(x_1 x_2)\psi_0^*(x_1, x_2') \qquad = \pi^{-\frac{1}{2}}(\gamma - \beta)^{-\frac{1}{2}} e^{\frac{-\gamma(x_2^2 + x_2'^2)}{2} + \beta x_2 x_2'} \tag{157}$$

The eigenvalues of the reduced density matrix ρ_{out} will turn out to be

$$\lambda_n = (1 - \xi)\,\xi^n. \tag{158}$$

where $\xi = \frac{\beta}{\gamma + \alpha}$. The formula for computing EE using these eigenvalues is known to be

$$S_{EE} = -\ln(1 - \xi_j) - \frac{\xi_j}{1 - \xi_j} ln\xi_j. \tag{159}$$

.

There is a lot of manipulation involving these parameters (β, ξ etc.), but if one pays attention, it can be realized that everything here is a function of the couplings k and

k_0.

In more general terms, consider a system of N coupled harmonic oscillators Eq.(160) in a given region. Such region will also be split into an inside sub-region, consisting of n oscillators, and an outside sub-region having N-m of such oscillators.

$$H = \frac{1}{2}\sum_{i}^{N} p^2 + \sum_{i,j}^{N} x_i K_{ij} x_j \tag{160}$$

In analogy with Eq.(156), we have the following groundstate wavefunction

$$\psi_0(x, ... , x_N) = \pi^{-N/4}(det\Omega)^{-1/4} e^{-\frac{x.\Omega.x}{2}} \tag{161}$$

To trace out the n inside variables, we use Eq.(162)

$$\begin{aligned} &\rho_{out}(x_{n+1}, ... , x_N; x'_{n+1}, ... , x'_N) \\ &= \int \prod_{i=1}^{n} dx_i \psi_0(x_1, ... , x_n; x_{n+1}, ... , x_N) \times \psi_0^*(x_1, ... , x_n; x'_{n+1}, ... , x'_N) \end{aligned} \tag{162}$$

The process is certainly more challenging. Following what (Srednicki, 1993) did, we define a block form $\Omega_{ij} = \sqrt{K}_{ij}$. that is composed of matrices that encompasses the oscillators of the inside sub-region (A) only, outside sub-region (C) only and mix both sub-regions (B) and (B^T).

$$\Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \tag{163}$$

The localized density matrix will be

$$\rho_{out}(x, x') \sim e^{-\frac{x.\gamma.x + x'.\gamma.x'}{2} + x.\beta.x'} \tag{164}$$

where

$$\beta = \frac{1}{2} B^T A^{-1} B \tag{165}$$

$$\gamma = C - \beta \tag{166}$$

It is known (Katsinis; Pastras, 2018) that the eigenvalues needed to compute the entropy

using the more general Eq.(167) ξ will be also the eigenvalues of $\gamma^{-1}\beta$

$$S_{EE} = \sum_{j=n+1}^{N} -ln(1-\xi_j) - \frac{\xi_j}{1-\xi_j} ln\xi_j \tag{167}$$

It is very difficult though to obtain the EE for the ground state because the system is highly entangled there. We would like to make a perturbative approach and this is possible in the case where the off-diagonal terms of Ω are much smaller than the diagonal ones.We will soon realize that this can be related to an inverse mass expansion. This expansion can be done from the 1+1 up to 3+1 dimensions and also from the first to the third order (as far as we are concerned here). It is chosen here to present briefly and partially the results for the 3+1 dimensions and third-order expansion.

Let's now take a free real scalar theory in 3+1 dimensions. The main idea is to modify the hamiltonian of this model to make it seem like a system of harmonic oscillators and use what we have already discussed. We can decompose it into spherical harmonics, this will give us $\phi_{lm}(r)$ and $\pi_{lm}(r)$ where l and m are obviously discrete. What about r? A lattice of spherical shells bounded by a spherical box may be introduced to take that into account. This will also impose UV and IR cutoffs, but we will not detail out that here. After all this manipulation, we get the following hamiltonian

$$\begin{aligned} H = \frac{1}{2a} \sum_{l,m} \sum_{j=1}^{N} [\pi_{lm,j}^2 + \left(j+\frac{1}{2}\right)^2 \left(\frac{\varphi_{lm,j+1}}{j+1} - \frac{\varphi_{lm,j}}{j}\right)^2 + \\ \left(l\left(\frac{l+1}{j^2}\right) + \mu^2 a^2\right) \varphi_{lm,j}^2], \end{aligned} \tag{168}$$

from which we obtain the the matrix K_{ij} and therefore Ω_{ij} and $\gamma^{-1}\beta$.

$$\begin{aligned} \mathrm{K}_{ij} &= K_i \delta_{ij} + (L_i \delta_{i+1,j} + L_j \delta_{i,j+1}) \\ \mathrm{K}_i &= \frac{ki^2}{\varepsilon^2} = 2 + \frac{l\,(l+1) + \frac{1}{2}}{i^2} + \mu^2 a^2 \\ \mathrm{L}_i &= L_i\,(k_i + k_{i+1}) \\ \mathrm{l}_i &= -\frac{\left(i+\frac{1}{2}\right)^2}{i\,(i+1)} \frac{1}{k_i + k_i + 1} \end{aligned} \tag{169}$$

The entanglement entropy for Eq.(168) at its groundstate will be given by Eq.(170). Note that we had to take into account a degeneracy in l.

$$S'_{EE} = \sum_{l=0}^{\infty}(2l+1)S_{EE} \tag{170}$$

We now will consider an entangling sphere that lies exactly between two neighboring lattice sites. This is useful because we will be able to associate an area term that appears in the inverse mass expansion to the entanglement entropy. Our entangling sphere radius is R and we define n_R.

$$n_R = n + 1/2 \tag{171}$$

$$R = n_R a \tag{172}$$

Eq.(170) cannot be calculated analytically, however, this series can be approximated by an integral using the Euler-Maclaurin method, where y is a change of variable to eliminate divergences.

$$S''_{EE} \cong n_R^2 \int_0^{\infty} dy S_l\left(N, n_R - \frac{1}{2}, y n_R^2\right) \tag{172}$$

By computing this integral for the third-order inverse mass expansion one obtains

$$\begin{aligned} S''_{EE} = & \\ & \left(\frac{3 + 2\ln\left[4 + (\mu^2 a^2 + 2)\right]}{16\left(\mu^2 a^2 + 2\right)} + \frac{167 + 492\ln\left[4 + (\mu^2 a^2 + 2)\right]}{4608\left(\mu^2 a^2 + 2\right)^3}\right. \\ & \left. + \frac{-11 + 2940\ln\left[4 + (\mu^2 a^2 + 2)\right]}{15360\left(\mu^2 a^2 + 2\right)^5} + (\mu^{-14})\right)\frac{R^2}{a^2} \\ & \approx 0,268\frac{R^2}{a^2} \end{aligned} \tag{173}$$

The results obtained previously via numerical calculation were $0.295\frac{R^2}{a^2}$ (Katsinis; Pastras, 2018).

A.2 Correlators Method

For our convenience, we can summarize the procedure of calculating EE for free theories using correlators. The system we will begin to examine is somehow generic, it consists of a set of harmonic oscillators embedded in a lattice.

Let us consider a region V and a class of generic operators O_V contained inside that region. Also, recall that one can describe a free field theory easily in terms of two point correlators via Wick's Theorem. That considered, the generic operators can be associated with generic correlators $\langle O_V \rangle$.

When the outside environment is traced out, we obtain a reduced density matrix for the inside region ρ_V. The way to relate the density matrix in question, the operators, and propagators is shown in Eq.(174).

$$\langle O_V \rangle = Tr \langle \rho_V O_V \rangle. \tag{174}$$

If we are dealing with bosons, the correlators we are interested in are the two-point functions of fields and conjugate momenta, namely $\langle \phi_i \phi_j \rangle = X_{ij}$ and $\langle \pi_i \pi_j \rangle = \Pi_{ij}$. And The density matrix has the ansatz expressed in Eq.(175). The multiplying factor $C = \Pi_l (1 - e^{-\epsilon_l})$ comes from normalization, and the form (Casini; Huerta, 2009) $e^{-\sum_l \epsilon_l a_l^\dagger a_l}$ is not a big leap considering we are dealing with bosons.

$$\rho_V = \Pi_l (1 - e^{-\epsilon_l}) e^{-\sum_l \epsilon_l a_l^\dagger a_l} \tag{175}$$

It is possible, of course, to relate our variables with the creation and annihilation operators Eq.(176) and Eq.(177) where f and p are matrices.

$$\phi = f_{ij}^* a_{ii}^\dagger + f_{ij} a_j \tag{176}$$

$$\pi = -ip^*_{ij} a^\dagger_i{}_i + ip_{ij} a_j \tag{177}$$

Applying Eq.(174) to the fields and momenta, we obtain

$$f^* n f^T + f(n+1) f^\dagger = X \tag{178}$$

$$p^* n p^T + p(n+1) p^\dagger = P \tag{179}$$

rewriting the ansatz in terms of fields

$$\rho_V = K e^{-\sum_v (M_{ij}\phi_i\phi_j + N_{ij}\pi_i\pi_j)} \tag{180}$$

where M and N are written in terms of Eq.(181) Eq.(182) and their square root Eq.(183)

$$M = \frac{1}{4} f^{-1} \epsilon f^{-1} = P \frac{1}{2C} log\left(\frac{C+\frac{1}{2}}{C-\frac{1}{2}}\right) \tag{181}$$

$$N = f \epsilon f^T = X \frac{1}{2C} log\left(\frac{C+\frac{1}{2}}{C-\frac{1}{2}}\right) \tag{182}$$

$$C = \sqrt{XP} \tag{183}$$

Then the entropy might be calculated via equation Eq.(184).

$$S = \sum_l \left(-log(1-e^{-\epsilon_l}) + \frac{\epsilon_l e^{-\epsilon_l}}{1-e^{-\epsilon_l}}\right) \tag{184}$$

One may read (Casini; Huerta, ; Schuch et al., 2006; Dutta et al., 1995) for more details about the above calculation .

APPENDIX B – Maximizing Shannon Entropy

Let us consider a classical variable X than can take n values $x_1, x_2, x_3...x_n$, each one with probabilitis $p_1, p_2, p_3...p_n$. The Shannon entropy $\mathcal{S}$ for this probability distribution is

$$\mathcal{S} = -\sum_i^n p_i \ln p_i. \tag{185}$$

Naturally, we have the following constraint concerning the probabilities

$$\sum_i^n p_i = 1. \tag{186}$$

Applying the method of Lagrange multipliers to Eq.(185) and Eq.(186), where we differentiate with respect to an arbitrary probability $p_1 \in p_i$ that we want to maximize

$$\begin{aligned} \frac{\partial}{\partial p_1}\left(-\sum_i^n p_i \ln p_i\right) - \lambda \frac{\partial}{\partial p_1}\left(\sum_i^n p_i\right) &= 0 \\ -\ln P_1 - 1 - \lambda &= 0 \\ p_1 &= e^{-(\lambda+1)} \end{aligned} \tag{187}$$

using Eq.(187) in Eq.(186), we have that the sum yelds n times p_1

$$\begin{aligned} \sum_i^n p_i = n e^{-(\lambda+1)} &= 1 \\ p_1 &= \frac{1}{n} \end{aligned} \tag{188}$$

So the entropy is maximized for equiprobable $\frac{1}{n}$. Substituting Eq(188) in Eq(185), finally proves that its value will be

$$\mathcal{S}_{max} = -\sum_i^n \frac{1}{n} \ln \frac{1}{n} = \ln n \tag{189}$$

APPENDIX C – Entropy for GHZ states

In this appendix, we show how to compute the EE for different partitions of the GHZ state $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$. The spins refer to subsystems 1,2,3 respectively. The density matrix for the entire state is therefore

$$\rho_{123} = \frac{1}{2}\left(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\right). \tag{190}$$

The subscripts 123 (together) is short for $1 \cup 2 \cup 3$.

We can find the reduced density matrix for subsystem 1

$$\rho_1 = \frac{1}{2}Tr_{23}\rho_{123} = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{191}$$

The remaining states, only belong to subsystem 1, despite it being omitted in the notation (which is quite common). For clarity, it means that, for example, the first combination of states $(|0_1 0_2 0_3\rangle\langle 0_1 0_2 0_3|$ that we had before in Eq.(190) was now reduced to $(|0_1\rangle\langle 0_1|)$ in Eq.(191). It follows that its entanglement entropy is

$$S_1 = \ln 2. \tag{192}$$

And it is easy to see that for this particular state

$$S_1 = S_2 = S_3, \tag{193}$$

as no subsystem is different from any other.

In a very similar way, we proceed to compute the EE for subsystem 12, which means we have to trace out the variables from 3, this is carried out like[9]

$$\rho_{12} = \frac{1}{2}Tr_3\rho_{123} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{194}$$

[9] Recall that, for example, $|000\rangle = \binom{1}{0} \otimes \binom{1}{0} \otimes \binom{1}{0} = \begin{pmatrix} \binom{1}{0}1 \\ \\ \binom{1}{0}0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

For this case, S_{12} is also easily calculated

$$S_{12} = \ln 2. \tag{195}$$

It is important to stress that this is a nice feature of the GHZ state and doesn't apply in general. The same entropy is obtained for combinations 23 and 13

$$S_{12} = S_{23} = S_{13}. \tag{196}$$

.

Finally, we can measure the entropy of the system as a whole, following the exact same procedures. We turn back to equation 190 and write it explicitly term by term in matrix form

$$\frac{1}{2}|000\rangle\langle 000| = \frac{1}{2}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}. \tag{197}$$

$$\frac{1}{2}|000\rangle\langle 111| = \frac{1}{2}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\end{pmatrix}. \tag{198}$$

$$\frac{1}{2}|111\rangle\langle 000| =$$

$$\frac{1}{2}\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}\begin{pmatrix}1&0&0&0&0&0&0&0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\end{pmatrix}. \tag{199}$$

$$\frac{1}{2}|111\rangle\langle 111| =$$

$$\frac{1}{2}\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}\begin{pmatrix}0&0&0&0&0&0&0&1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\end{pmatrix}. \tag{200}$$

$$\rho_{123} = \frac{1}{2}\begin{pmatrix}1&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&1\end{pmatrix}. \tag{201}$$

Because the only non zero eigenvalue of Eq.(201) is $\lambda = 1$,

$$S_{123} = -\lambda ln\lambda = -\ln 1 = 0 \tag{202}$$

This is expected due to the total density matrix being pure[10]. So any partitions are maximally entangled with each other but the system as a whole is not.

[10] In contrast, a mixed density matrix has the form $\rho = \sum_{i=1}^{n} p_i \left|\psi\right\rangle \left\langle\psi\right|$.

APPENDIX D – Expansion of EE

Starting from the expression in Eq.(148)

$$S_{EE} = \frac{\pi}{3}\text{Vol}^{(\mathbb{R}^{D-2})}\left(\frac{m}{2}\right)^D \frac{\epsilon^D}{m^D} e^{-\frac{1}{\epsilon^2 m^2}} \sum_{n=0}^{\infty} \frac{(D/2)_n}{n!}\left(\frac{\epsilon^2}{m^2}\right)^n, \tag{203}$$

we can use the definition of the Pochhammer symbol

$$(D/2)_n = \frac{D}{2}\cdot\left(\frac{D}{2}-1\right)\cdots\left(\frac{D}{2}-(n-1)\right) = \frac{D}{2}\cdot\frac{(D/2-1+n)!}{(D/2-1)!} \tag{204}$$

in the original expression

$$S_{EE} = \frac{\pi}{3}\,\text{Vol}^{(\mathbb{R}^{D-2})}\left(\frac{m}{2}\right)^D \frac{\epsilon^D}{m^D} e^{-\frac{1}{\epsilon^2 m^2}} \sum_{n=0}^{\infty} \frac{1}{n!}\frac{D}{2}\frac{(D/2-1+n)!}{(D/2-1)!}\left(\frac{\epsilon^2}{m^2}\right)^n. \tag{205}$$

Next, we can Taylor expand the exponential

$$e^{-1/(\epsilon^2 m^2)} = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{\epsilon^2 m^2}\right)^n}{n!}. \tag{206}$$

to finally obtain

$$\begin{aligned} S_{EE} &= \frac{\pi}{3}\,\text{Vol}^{(\mathbb{R}^{D-2})}\left(\frac{m}{2}\right)^D \frac{\epsilon^D}{m^D} \sum_{n=0}^{\infty} \frac{1}{n!}\frac{D}{2}\frac{(D/2-1+n)!}{(D/2-1)!}\frac{1}{n!}\frac{\left(-\frac{1}{\epsilon^2 m^2}\right)^n}{n!}\left(\frac{\epsilon^2}{m^2}\right)^n \\ &= \frac{\pi}{6}\frac{\text{Vol}^{(\mathbb{R}^{D-2})}}{(4\pi)^{D-1}}\left[\frac{1}{(D-2)\epsilon^{D-2}} - \frac{m^2}{(D-4)\epsilon^{D-4}} + \frac{m^4}{8(D-6)\epsilon^{D-6}} - \cdots\right] \end{aligned} \tag{207}$$

as we wanted to demonstrate.

APPENDIX E – Bipartion in relativistic field theory

In a Relativistic Quantum Field Theory (QFT) in D-dimension, the model is formulated on a background spacetime manifold $\mathcal{M}$ with a causal domain $D^{\pm}(R)$, where the $\pm$ signs correspond to the past and future. The theory works well in a globally hyperbolic spacetime manifold, where causal connections between points are well-defined. Such a spacetime manifold admits a Cauchy slice Σ, which is a spacelike hypersurface that can be used to define the entire causal domain, given by

$$D^{+}[\Sigma] \cup D^{-}[\Sigma] = \mathcal{M}. \tag{208}$$

The domain of dependence $D[A]$ is defined as the set of points in B, where A is intersected by every causal curve that cannot be extended(Rangamani; Takayanagi, 2017). It is the region in which the reduced density matrix can be evolved. The domain of dependence is not necessarily the same as the causal domain

$$D^{+}[A] \cup D^{-}[A] \neq \mathcal{M}. \tag{209}$$

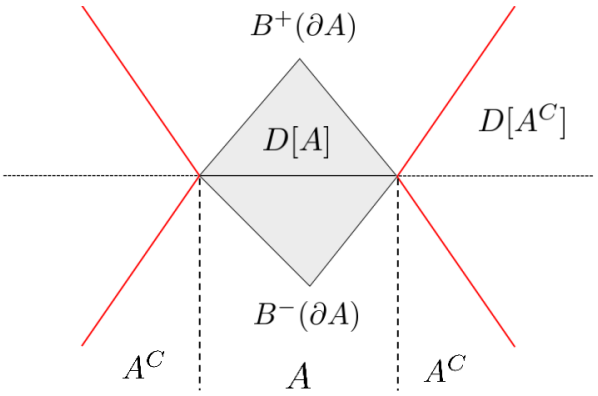
So we have to take into account the regions that are on the boundary, in the sense that are not int A or A^{C}. This is usually represented by the "causal diamond" of the region and its complement Fig(9).

$$\mathcal{M} = D[A] \cup D[A^{C}] \cup B^{+}(\partial A) \cup B^{-}(\partial A) \tag{210}$$

where $B^{+}(\partial A)$ and $B^{-}(\partial A)$ of the points that are neither in A or A^{C} .

The relevance of this construction lies in the fact that the reduced density matrices ρ_A, as well as the corresponding entropy, are associated not with the Hamiltonian of the subregion $\mathcal{H}_A$, but with the causal domain$D[A]$. Thus, even if we take a different region and Cauchy slice, if their domains of dependence are the same, both the reduced matrices and the entropy will be the same, that is $D[A] = D[C] \Rightarrow S[A] = S[C]$.

Figure 9 - Causal diamond



Legend: the subregions A and A^C and their domains of dependence.
Source: The author, 2023.