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Foundational Studies in Proof-theoretic Semantics

Rio de Janeiro
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Tese apresentada, como requisito parcial para obtenção do título de Doutor, ao Programa de Pós-graduação em Filosofia, da Universidade do Estado do Rio de Janeiro. Área de concentração: Filosofia.

Orientador: Prof. Dr. Luiz Carlos Pinheiro Dias Pereira

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INSCRIPTION

To my Clarinha.

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"Don't worry: it's always possible to prove the opposite"

"It is very common for an unbelievable truth to transform into an easily assimilable lie"

Millôr Fernandes. Millôr Definitivo – A Bíblia do Caos.
Entries "Ângulo" and "Verdade", freely translated from Portuguese.

ABSTRACT

BARROSO NASCIMENTO, V. L. *Foundational Studies in Proof-theoretic Semantics*. 2024. 138 f. Tese (Doutorado em Filosofia) – Instituto de Filosofia e Ciências Humanas, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, 2024.

This thesis investigates the technical and conceptual foundations of multibase semantics, a new kind of proof-theoretic semantics. Proof-theoretic semantics are frameworks in which the tools of Proof Theory are used for the semantic analysis of logics, bridging the gap between formal syntax and semantics. The first part of the thesis focuses on conceptual aspects of the notions of truth and proof, arguing that their distinct philosophical characteristics naturally lead to differences in their formal characterizations. It is also argued that, even though such proposals are usually presented by defenders of intuitionistim, proof-theoretic semantics should not be made for the intuitionist alone. The second part focuses on the technical aspects of multibase semantics. Multibases are presented in a standard and a focused version; standard multibases are no different from Kripke models for minimal logic, but focused multibases are shown to have many other interesting properties. In particular, focused multibases allow a generalization of the notion of S-validity, one of the main selling points of another semantics called proof-theoretic validity. Proof-theoretic validity, originally proposed by Prawitz and later championed by Dummett, was one of the first proposed proof-theoretic semantics, but the interest initially surrounding it was partially lost after a plethora of negative results were discovered (including incompleteness ones). Generalized S-validity is shown to have almost all of the properties originally expected to hold for S-validity, including completeness with respect to minimal logic. In particular, generalized S-validity is shown to be completely reducible to atomic derivability, and it is shown that multibases for predicate logic can be obtained without the aid of any model-theoretic notions. We also show that, as is expected of proof-theoretic semantics, it is possible to use methods characteristic of Proof Theory to obtain results that are semantic in nature.

Keywords: proof-theoretic semantics; base-extension semantics; proof-theoretic validity.

RESUMO

BARROSO NASCIMENTO, V. L. *Estudos Fundacionais em Semânticas Prova-teóricas*. 2024. 138 f. Tese (Doutorado em Filosofia) – Instituto de Filosofia e Ciências Humanas, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, 2024.

O objetivo desta tese é investigar os fundamentos técnicos e conceituas da semântica de multibases, uma nova modalidade de semântica prova-teórica. Semânticas prova-teóricas são aquelas nas quais se busca analisar aspectos semânticos de lógicas através do uso de ferramenas típicas de Teoria da Prova, promovendo uma aproximação entre sintaxe e semântica formal. A primeira parte da tese aborda aspectos conceituais das noções de prova e verdade, argumentando que diferenças filosóficas entre ambas naturalmente levam a diferenças em suas caracterizações formais. Também é apontado que, embora as propostas prova-teóricas frequentemente sejam apresentadas por defensores do intuicionismo, elas não deveriam se restringir apenas a este público. A segunda parte aborda aspectos técnicos da nova semântica. Multibases possuem uma versão padrão e uma versão focada; multibases padrão não são significativamente diferentes de modelos de Kripke para a lógica minimal, mas multibases focadas possuem muitas propriedades adicionais interessantes. Em particular, multibases focadas nos permitem generalizar a noção de S-validade, um dos principais conceitos da abordagem semântica denominada validade prova-teórica. Originalmente proposta por Prawitz e posteriormente capitaneada por Dummett, a validade prova-teórica é considerada uma das principais propostas apresentadas durante a gênese das semânticas prova-teóricas, mas o interesse nela se esvaiu com a descoberta de uma série de resultados negativos (inclusive resultados de incompletude). Nós demonstramos que a noção de S-validade generalizada possui quase todas as propriedades esperadas da noção original de S-validade, o que inclui resultados de completude para a lógica minimal. Dentre os principais resultados, destacamos os que mostram a total redutibilidade da S-validade generalizada à noção de demonstrabilidade atômica, bem como os que mostram a possibilidade de extensão da semântica para as lógicas de predicados sem o auxílio de qualquer ferramenta modelo-teórica. Também demonstramos que, como seria esperado de uma semântica genuinamente prova-teórica, é possível usar ferramentas típicas de Teoria da Prova para provar resultados de natureza semântica.

Palavras-chave: semânticas prova-teóricas; semânticas de extensão de base; validade prova-teórica.

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INTRODUCTION

Mathematization wrought, at the same time, the greatest blessing and direst curse to ever befall logic. The precise and orderly nature of formal methods granted logicians the ability to repeatedly temper theories until their strongest versions are reached, while at the same time distancing them from the very intuitions that led to formalization. As eloquently put by Dana Scott, “it is all too tempting to refine methods well beyond the level of applicability. The puzzle is the opiate of the thinker” (SCOTT, 1973). The emancipation of logical form from logical matter inevitably made possible both the study of formalism for its own sake and the creation of *post hoc* intuitions for systems of logic, two unfortunate paths many contemporary logicians choose to take.

It is in this vein that Dirk van Dalen opens the preface of his book “Logic and Structure” by stating that:

> Logic appears in a “sacred” and in a “profane” form; the sacred form is dominant in proof theory, the profane form in model theory.

Sacrality is nothing more than a shield against the emancipation from intuition that profanity brings about, and also from the sins it may lead to (in the shape of antinomies and paradoxes). Proof theory allows us to remain true to our mission by promoting the safe development of intuitions through mathematical restraint, whereas model theory gives us powerful wings that do not fare well if one gets too close to the sun.

The purpose of this thesis is to further advance the study of proof-theoretic semantics, an approach in which the tools of proof theory are used to deal with a subject traditionally analyzed through the lens of model theory. Ideally, this will lead to a framework in which sacred methods lay solid grounds for activities requiring profane powers, in the hopes that this yields something with the solidity and simplicity usually observed in proof theory and the power and generality usually observed in model theory.

The first chapter lays out the philosophical grounds of our work. Its discussions range from (confessedly idiosyncratic) topics in philosophy to important notions that will later lend themselves to formalization. Particularly important are the discussions on the concepts of proof and truth, as the main philosophical difference between model-theoretic and proof-theoretic semantics is which notion is taken as fundamental. Since Proof Theory is often still associated with syntactic methods, we also discuss what makes a definition semantic and how there can be a semantics in terms of proofs.

The second chapter defines basic technical notions and provides model-theoretic semantic definitions, later to be compared to their proof-theoretic counterparts.

The third and final chapter presents technical results and philosophical discussions on a new kind of proof-theoretic semantics called multibase semantics. This new semantics is very similar to Prawitz and Dummett's approach to proof-theoretic validity, and it also allows a generalization of their concept of S-validity. Almost all of the desiderata originally listed for S-validity are shown to hold for generalized S-validity, including soundness and completeness with respect to minimal and intuitionistic logic, complete reducibility of validity to atomic derivability, and validity of Export. Multibases can also be used to deal with predicate logic without the aid of any model-theoretic notions. Since domains and interpretations are defined entirely through proof-theoretic means, predicate generalized S-validity is still completely reducible to atomic derivability, and most of the results for propositional multibases can be smoothly extended to first and second-order multibases.

1 PHILOSOPHICAL DISCUSSIONS

1.1 Preliminary discussions

1.1.1 The axiomatic nature of philosophy

The starting point of any philosophical investigation must be the personal perspective of an acting subject. Breaking through a initial moment of inertia, the subject reflects about his own perceptions in order to attain a theoretical understanding of reality. Provided he also inquires about his own philosophizing, the subject either recognizes his active subjecthood or incurs a contradiction, since in denying it he would be exercising the same faculties he denies having. The subject can thus neither deny his existence nor deny his activity, since only an active, existing being is capable of denial. It is with this insight that Descartes laid the groundwork for modern philosophy in his Meditations (DESCARTES, 2008), a magnanimous contribution to philosophy that also sets a very convenient starting point for any inquiry on its nature.

Although Descartes proceeds to construct a very particular *corpus* from his original insight, one does not need to be a full-fledged Cartesian to recognize active subjecthood as a necessary condition for philosophical activity. A subject can, for example, assert his own active subjecthood while also rejecting Descartes' proofs of the existence of an external world. This is so because Descartes relies on additional insights to develop his ideas, the acceptance of which is quite independent from that of the first one.

The initial insight guarantees only that a philosopher must either recognize his own active subjecthood or contradict himself. It does not prevent him from failing to pay heed to the warning and denying his own ability to deny. It does not show, absent other insights, to what direction should philosophical inquiry be conducted. It also does not prevent the philosopher from returning to his inertia and abstaining from any further inquiries. It merely sets up, to borrow an expression from Dummett (DUMMETT, 1991, pg. 16), a base camp for conducting a non self-contradictory philosophical assault. The foundational act of recognition does provide us with a solid foundation for the construction of doctrines, but it does not determine any philosophical guideline or endpoint.

From each of his own insights[1] the philosopher may extract one or more *philosophical axioms*[2], which for convenience may be divided into *static* and *dynamic axioms*. Static axioms dictate what is axiomatically accepted, whereas dynamic axioms dictate which methods

[1] Insights are behavioral in nature and their content is primitive, in the sense of not being definable in terms of any constituents.

[2] Due to their foundational nature, we could say that the content of a philosophical axiom is what Wittgenstein calls a hinge proposition (WITTGENSTEIN, 1972, pg. 44).

capable of leading to the acceptance of non-axioms are axiomatically accepted. When a sufficient amount of axioms is established, the philosopher may produce arguments and establish a complete philosophical doctrine.

A natural conclusion of this characterization is that any philosophical doctrine must have as its foundation those axioms, which are based only on the subject's insights[3]. Since they are nothing more than constructions based on subjective experiences, they have no extra-subjective validation. But, and this is the *crux* of philosophical disputes, the philosopher may perceive the possibility of different insights and yearn for a stronger kind of validation, hoping for there to be some absolute criterion allowing him to differentiate between correct and incorrect axioms. Sadly, when he tries to do so, he must again rely on axioms that classify other axioms into correct and incorrect, so he perpetuates his self-grounding dilemma. Unfortunately, this means that philosophy must be characterized as a deeply personal, profoundly dogmatic, and inescapably idiosyncratic subject.

All is not lost. Provided a subject perceives something akin to an external world and entities akin to independent subjects, philosophy may start to be characterized as a social activity. The subject does not even need to concede the externality of those perceptions, as they are already sufficient for the justification of his activities. He may engage those perceptions with the intent of refining his axioms, leading to new doctrinal developments and, incidentally, to the furthering of the subject's sense of knowledge. Even though this framework makes all knowledge uncertain, the philosopher might find it quite therapeutic to entertain the impression that he is peering into the nature of reality.

A philosopher who adopts this perspective and wishes to be honest must then admit the absolute subjectiveness of his own insights, and thus of his axioms. Ideally, this should not discourage him from presenting the reasoning behind his doctrine as best as he can, either because he draws satisfaction from the sensation of obtaining knowledge or because he personally believes in the externality of his perceptions and wishes to collectively reach a better sense of understanding.

[3] Though our phrasing might suggest otherwise, we are not adopting the dogmatist solution to Agrippa's trilemma. According to the trilemma, which draws upon three of Agrippa's five modes (WHITE, 2021, pgs. 399-400)(ANNAS; BARNES, 2000, pgs. 40-43), anyone attempting to justify a claim should also justify the justification, and this leads to one of three issues: either some justification is accepted without any further justification (mode by hypothesis), every justification is given a new justification in a infinite chain (mode of endless regress) or some justification is circularly justified by something it justifies (mode via reciprocals). This arguments is used by Agrippa in defense of Phyrronian skepticism (FOGELIN, 1994). The trilemma is only indirectly related to our account because it concerns how accepted claims are justified, whereas we only deal with the act of acceptance itself.

1.1.2 Consequence relations and the definition of logic

There are many perspectives from which one can obtain an insight of what ought to be the definition of *logic*. Such a definition can be produced from the internal perspective of any philosophical doctrine, and a robust doctrine naturally yields a substantive but partisan notion. One philosopher might give a definition which leads to the conclusion that classical logic is the one true logic, and another might do just the same with a definition justifying intuitionistic logic. An empiricist such as Mill might give an account based on empiricist axioms (MILL, 1874), and a platonist such as Frege might rebuke it based on platonist axioms (FREGE, 1958). Our earlier considerations may be taken to dispel any illusion of giving a properly agnostic, doctrine-independent definition, since definitions are always given in terms of axioms. But there is still an important choice to be made: we can either give a robust definition, based on many philosophical axioms, or a minimalist definition, based on only a few.

A definition grounded on few axioms may be used together with new axioms to obtain a new definition, which will then be grounded both on the new axioms and on those of the first definition. As such, for any *corpus* we can construct an ordering of definitions in terms of how many axioms they presuppose: the greater the number, the less *basic* the notion. The most basic notions of a doctrine are usually regarded as their foundations, since they provide justification for more complex (and usually less intuitive) notions. Thus, in order to choose between robust and minimalist definitions, we must take into account how basic we want our notion of logic to be.

If we intend to give a definition of logic which is taken to explain or justify the validity of arguments in general, the insight that it must be a very basic notion seems quite reasonable. In fact, the notion of logic seems to be a prerequisite for valid argumentative justification of any philosophical claim[4]. Whenever we argue in favour of something instead of just taking it for granted, we are already depending on some notion of what constitutes a valid argument – however vague, informal, or unstated. Since an axiomatic definition of what counts as a valid argument is a prerequisite for the justification of any philosophical claim, it seems desirable that a definition of logic be given as soon as we wish to produce justified claims instead of just creating new axioms. Hence, as long as we want as many claims as possible to be justified and argued for, it is only natural that we consider logic to be one of the most basic notions.

Following this insight, an elegant and quite basic definition which may be adopted is the following (BEALL; RESTALL, 2006, pg. 3):

> Logic is about consequence. Logical consequence is the heart of logic; it is also at the centre of philosophy and many theoretical and practical pursuits besides. Logical consequence is a relation among claims (sentences, statements,

[4] Prawitz applies similar reasoning to the notions of argument and validity (PRAWITZ, 2012, pg. 2).

> propositions) expressed in a language. An account of logical consequence is an account of what follows from what—of what claims follow from what claims (in a given language, whether it is formal or natural). An account of logical consequence yields a way of evaluating the connections between a series of claims—or, more specifically, of evaluating arguments.

It is possible to axiomatically establish, borrowing this definition, that a logic is an *account of relations of logical consequence between statements*. We take for granted both that we have a language in which statements can be expressed and that there can be relations between them, and then stipulate criteria according to which some of the latter are considered to be relations of *logical* consequence. As such, although quite basic, this notion is still restricted by the constraint of *logicality*.

This narrows down the basics, but also leaves open an important question. What is logical consequence, and which requirements a consequence relation must fulfill in order to be logical? Those requirements are usually imposed in order to separate logical consequences from those of another kind, such as material consequences. Some properties, such as *necessity* and *formality*, are commonly listed as basic desiderata (BEALL; RESTALL; SAGI, 2019). The literature contains many debates on the subject, but an overarching agreement on the essential properties of logicality has yet to be established[5].

An insight which might serve those who want the most basic definition of logic is that logicality constraints fulfill the purpose of *negative criteria*, that is, they exclude a subset of the set of all consequence relations from the set of properly logical relations. While those restrictions might be philosophically interesting, the notion of logical consequence obtained through such a constraint is strictly less basic than an unristricted notion encompassing all consequence relations. From this perspective, the most basic notion of logical consequence is one in which *no logicality constraint is imposed at all*.

This unrestricted notion of logical consequence may be seen as one which takes to heart Carnap's Principle of Tolerance (CARNAP, 2014, pg. 52):

> *In logic, there are no morals*. Everyone is at liberty to build up his own logic, i.e. his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments.

[5] "If one wants to know whether there is unanimity or disagreement among the experts concerning the basic tenets of a particular subject, one way to find out is to look at what they say when introducing their subject to the uninitiated. Applying this approach to logic, one is tempted to conclude that the foundations of the subject must be in disarray. An examination of respected texts written by established practitioners reveals considerable disagreement about the pre-theoretic notion of logical consequence. Furthermore, these texts usually do not mention this disagreement. It is as though their authors either haven't noticed it or don't recognize its importance" (HANSON, 1997, pg. 366).

This position will undoubtedly be considered outrageous by those with more robust insights on the matter, but it must be pointed out that it is entirely possible to conduct logical investigations without pledging alliance to any notion of logicality. In fact, if we are inclined to consider that two people operating under incompatible logicality notions may both be doing logic, we are implicitly using a definition that is independent of logicality. Although the study of formal systems deemed adequate by some demarcation may be of great philosophical interest, there is no reason for us to choose a constraint for the delimitation of the study of logic itself[6].

Tarski seems to have faced essentially the same issue when trying to draw a line between logical and non-logical constants (TARSKI, 1983, pgs. 418-420). When discussing how to define which constants should be regarded as logical for the purpose of defining a *formal* notion of consequence, he writes:

> On the other hand, no objective grounds are know to me which permit us to draw a sharp boundary between the two groups of terms. It seems to be possible to include among logical terms some which are usually regarded by logicians as extra-logical without running into consequences which stand in sharp contrast to ordinary usage. In the extreme case we could regard all terms of the language as logical. The concept of *formal consequence* would then coincide with that of *material consequence*. (...) Perhaps it will be possible to find important objective arguments which will enable us to justify the traditional boundary between logical and extra-logical expressions. But I also consider it to be quite possible that investigations will bring no positive results in this direction, so that we shall be compelled to regard such concepts as 'logical consequence', 'analytical statement', and 'tautology' as relative concepts which must, on each occasion, be related to a definite, although in greater or less degree arbitrary, division of terms into logical and extra-logical.

By establishing that logic is an account of consequence relations in general, we allow many distinct notions of logicality to flourish under it, since an account of consequence relations informed by a particular notion of logicality is still an account of consequence relations (and thus a logic). The notion can be viewed as basic precisely because it does not exclude any particular definition of logicality from the field of logic[7].

In order to further support this point, perhaps the most controversial of this section, a quick analogy can be made to show the usefulness of such a broad conception.

One of the oldest debates in the field of Philosophy of Law is the one concerning basic moral requirements a law should fulfill in order to be considered a proper legal rule. Some philosophers, most notably Hans Kelsen (KELSEN, 1967) and H. L. A. Hart (HART, 2012),

[6] This may be characterized as a position in which logic is viewed as a *method*, not as a *subject matter*, a distinction entertained in (MACFARLANE, 2000.).
[7] This position is very similar to the one described as *Relativist* in (MACFARLANE, 2017).

adopt some kind of basic notion in which no morality requirements are imposed upon the content of laws[8]. Other philosophers, such as Gustav Radbruch, argue for the existence either of an idea of "natural law" or of a binding idea of *justice*, which precedes man-made "positive laws" and morally constrains them (RADBRUCH; PAULSON; PAULSON, 2006). A positive law is thus considered a proper legal rule only if it does not conflict with basic moral contents of the natural laws or of the idea of justice. As such, those philosophers characterize extremely immoral state-enacted legislation to be not an example of "immoral law", but an example of "absence of law"[9]. The first school of thought is usually called *legal positivism*, the second one *jusnaturalism*.

Many objections can be levied against the jusnaturalist conception, but one is of special interest to us: if we consider extremely unjust laws to be non-laws (that is, positive laws which are not legal rules) instead of just immoral laws, we run the risk of seeing "non-laws" applied in a way which is functionally indistinguishable from the application of proper laws. A non-law enacted in a dictatorship could empirically (albeit immorally) be enforced with the aim of seizing property of oppressed minorities just as much as a proper law could be enforced with the aim of seizing property of a debtor in a less unjust system. The distinction between law and non-law is irrelevant to the subject of the State with respect to practical consequences of the law, thus the difference between positive laws and proper legal rules is irrelevant to anyone but the legal theorist[10].

We claim that the relation between *positive law* and *natural law* is of the same nature as the relation between *consequence* and *logicality*. Logicality is just as contingent to the basic characterization of logic as morality is to the basic characterization of law. In fact, there seems to be as much disagreement between logicians on the contents of logicality as there is between practitioners of law on the contents of morality. By introducing morality into the concept of law and logicality into the concept of logic, we are guaranteeing that laws are always moral and logics always well-behaved, but at the cost of ignoring the existence of many "non-laws" and

[8] "The thesis, widely accepted by traditional science of law but rejected by the Pure Theory of Law, that the law by its nature must be moral and that an immoral social order is not a legal order, presupposes an absolute moral order, that is, one valid at all times and places. Otherwise it would not be possible to evaluate a positive social order by a fixed standard of right and wrong, independent of time and place" (KELSEN, 1967, pg. 68).

[9] "One line of distinction, however, can be drawn with utmost clarity: Where there is not even an attempt at justice, where equality, the core of justice, is deliberately betrayed in the issuance of positive law, then the statute is not merely 'flawed law', it lacks completely the very nature of law. For law, including positive law, cannot be otherwise defined than as a system and an institution whose very meaning is to serve justice. Measured by this standard, whole portions of National Socialist law never attained the dignity of valid law" (RADBRUCH; PAULSON; PAULSON, 2006, pg. 7).

[10] The distinction could still be, and usually is, rhetorically useful to lawyers whenever the positive law (or even legal practice, considered in a broad sense) imposes morality constraints on the application of laws, but this can happen only when those criteria of morality are somehow positively incorporated into the legal system. But this discussion is, of course, entirely outside the scope of this thesis.

"non-logics" with the same effects of their proper counterparts[11]. Morality and logicality have been on the spotlight for so long and present themselves to us so frequently that this important conceptual distinction may not be immediately intuitive, but it is an important distinction nevertheless.

We need not consider all distinct logics to be equally interesting, but we also need not deny their character as proper logics. Just as an extremely unjust legal order should still be considered a (absolutely immoral) legal order, an extremely useless logic should still be considered a (absolutely uninteresting) logic. When specifying some property of consequence relations considered of particular philosophical interest, we are merely specifying which subset of the set of all possible logics we are interested in.

1.1.3 Evaluations and metalanguages

A logic is defined through an *account* of consequence relations, that is, through the specification of a formal method for evaluating and classifying consequence relations between statements of a language. This is done by fixing both a collection of *evaluation categories*, in terms of which consequence relations between particular statements (or collections of statements) are to be judged, and a collection of *evaluation criteria*, in terms of which such judgements shall be dispensed. In other words, evaluation categories are the values assigned (implicitly or explicitly) to consequence relations, and evaluation criteria specify how these assignments are made.

Many distinct possibilities become available after we do away with the moral requirements of logicality, as any collection of categories and criteria may be taken as sufficient for defining a particular logic. Nevertheless, even in the context of non-classical logics, it is quite usual for a logic to be defined through a *binary* collection of evaluation categories, which separates consequence relations into *valid* and *invalid*, and some collection of criteria which is *total*, in the sense that it assigns either the value "valid" or the value "invalid" to all consequence relations on the appropriate language.

Once the categories and criteria are determined, both the logic itself and its *metalanguage* have been defined. The logic is defined through the value assignments made to each consequence relation, and the metalanguage through both the choice of evaluating categories and structural properties of assignment procedures determined by the criteria. If in some account we observe that the consequence relations have behaviors expected of intuitionistic logic but that it has two evaluation categories and criteria which determine procedures with structural

[11] For many examples of (modal) logics which would be considered non-logics by some due to not being closed under uniform substitution, see the first footnote of (HOLLIDAY; HOSHI; ICARD III, 2013).

properties characteristic of classical logic, we may conclude that it is a definition of intuitionistic logic through a classical metalanguage.

Curiously enough, a particular logic is usually identified only through the behavior of its object language, even though it is impossible to define an object language without also defining a metalanguage. This allows the same logic to receive very distinct *characterizations*. Not only are we allowed to use different metalanguages in order to characterize the same logic, but also to give different collections of categories and criteria which, in the end, will amount to different descriptions of the same logic. Some of those differences have important and far-reaching consequences, such as the one between *syntactic* and *semantic* specification procedures (to be examined in more depth later).

1.1.4 Inferences, composition and arguments

When defining the criteria which determine how categories are assigned to consequence relations, we can either provide a *total specification*, in which we effectively make a list of which relation belongs to which category, or a *procedural specification*, in which we define a list of basic procedures for evaluating consequences and use them to define which relations belong to which category. Since total specification is quite impractical, logics are usually defined through procedural specification.

For convenience, we may define procedural specification by recourse to two kinds of basic units. The first one is what is usually known as an *inference*, which amounts to a direct evaluation of a consequence relation between a linguistic object or a collection[12] of linguistic objects Γ and a linguistic object or collection of linguistic objects Δ according to an evaluation category c. It may be graphically represented as follows:

$$c\,\frac{\Gamma}{\Delta}$$

This inference evaluates as c the consequence relation between the object/collection Γ and the object/collection Δ. References to the category c may naturally be omitted whenever inferences are defined only for one category. Notice also that collections of collections of linguistic objects may also be considered linguistic objects themselves.

The second basic unit, which we may call *reasoning procedures*, consists in any procedure specifying how linguistic elements and inferences may be used to evaluate consequence relations. Unlike inferences, which directly specify the evaluation of a consequence, reasoning

[12] "Collection" is used here in a very broad sense: we can use sequences, sets, multisets or any other formal way of agglomerating linguistic elements.

procedures have as their output the evaluation of consequence relations[13]. If we take natural deduction as an example, the notions of self-deduction (any assumption A can be taken as a deduction of A from A) and composition (deductions can be used together with inferences to obtain new deductions) may be taken to be reasoning procedures, as they specify how deductions showing the validity of a consequence relation can be obtained through the use of inferences and the language.

Once a collection of basic units of specification has been established, we can define the notion of *argument* for a given logic by recourse to combinations of inferences and reasoning procedures. Arguments are structures obtained through such procedures with the purpose of evaluating a consequence relation between a first object/collection Γ (the argument's premises) and a second object/collection Δ (the argument's conclusion). The evaluation criteria usually dictate that the existence of an argument establishes validity for a consequence relation between its premises and its conclusion, and also that any relation for which an argument cannot be given is considered invalid.

Two aspects of the previous definitions are worthy of comment.

First, we should specify that we are dealing with a basic notion of inference, not a robust notion of *valid inference*. Inferences are usually presented as the basis of argumentation in general, whilst valid inferences are presented as the basis of logical argumentation in particular (PRAWITZ, 2019b) (PRAWITZ, 2012). But validity-based notions are too restrictive for those who wish to regard the notion of inference as basic, that is, to adopt a notion as tolerant as our notion of logic. The relation between validity and inference is of the same nature as the relation between logicality and consequence, so validity must be kept separate from inference if we desire our notions to be as basic as possible.

Second, even though our presentation is graphically similar to the one usually seen in natural deduction and other proof-theoretic systems, we aim to give a much broader definition of procedural validity. Consider, for example, the clause for conjunction in usual model-theoretic semantics:

$$v(A \land B) \Longleftrightarrow v(A) = 1 \text{ and } v(B) = 1.$$

If we take valuation functions to be linguistic objects and the semantics to be two-valued (with values 1 and 0), this can be interpreted as a simultaneous definition of the following seven inferences for every valuation v:

$$\frac{v(A \land B) = 1}{v(A) = 1} \qquad \frac{v(A \land B) = 1}{v(B) = 1} \qquad \frac{v(A) = 1 \qquad v(B) = 1}{v(A \land B) = 1}$$

[13] As noted before, the distinction is merely conventional: inferences can be defined as special cases of reasoning procedures if we consider direct specification to be a special kind of procedure.

$$\frac{v(A \wedge B) = 0 \qquad v(A) = 1}{v(B) = 0} \qquad \frac{v(A \wedge B) = 0 \qquad v(B) = 1}{v(A) = 0}$$

$$\frac{v(A) = 0}{v(A \wedge B) = 0} \qquad \frac{v(B) = 0}{v(A \wedge B) = 0}$$

We also include in our set of procedures one (and only one), of the two following inferences, for every v and every A:

$$\frac{}{v(A) = 1} \qquad \frac{}{v(A) = 0}$$

The first definition can clearly be replicated for all other connectives according to usual model-theoretic definitions. Combining those with the usual notion of deduction defined in natural deduction (PRAWITZ, 2006), truth for a formula A in a particular model v can be reduced to the existence of a deduction of $v(A) = 1$, and falsity for A in v to the existence of a deduction of $v(A) = 0$. Validity for the semantic consequence relation $\Gamma \vDash A$ between a set of formulas Γ and a formula A can then be defined through a broader evaluation criterion as holding whenever an argument for the truth of A can be given in every v for which an argument can be given for the truth of all formulas in Γ.

We thus argue that inferences and reasoning procedures may be considered basic units of evaluation even in frameworks such as model-theoretic semantics[14], although the notation used here is confessedly more convenient for proof-theoretic projects.

1.2 **Truth and Proof**

Up until now our main goal was to use as few philosophical axioms as possible in order to define basic notions. We will now impose additional constraints on those to obtain some robust notions that are deemed of interest.

1.2.1 Property preservation

As mentioned before, inferential definitions of logic are often obtained by using some notion of validity to constrain the notion of inference, which then yields some concept of *valid*

[14] A similar argument can be seen in (MARTIN-LÖF, 1987).

inference. Naturally, a robust account of inferential validity must be grounded in some philosophical notion that allows us to explain what makes an inference valid. One way to do so is by defining validity in terms of *property preservation*: an inference is *valid* if, whenever the premises of the inference have some property, it follows that the conclusion of the inference necessarily has that property[15]. Likewise, a *valid argument* may be defined as an argument obtained through the use of *valid reasoning procedures*, that is, procedures which guarantee that, if all of the argument's inferences are valid, then the property is transmitted from the argument's premises to its conclusion. Inferences are then regarded as valid due to their ability to transmit the chosen property from its premises to its conclusion, and reasoning procedures are valid due to their ability to allow the construction of valid arguments from valid inferences.

Despite its robustness, the definition of validity through property preservation has no particular philosophical affiliation. Although it is traditionally used by model-theoretic semantics to define inferential validity as the necessary preservation of truth (DOGRAMACI, 2015), it has also been used in the context of proof-theoretic semantics to define inferential validity as the necessary preservation of *proofs*, *grounds* or *justifications* (PRAWITZ, 2012) (PRAWITZ, 2019b). Since the property to be preserved is not defined beforehand, the notion may be used for the definition of many distinct notions of validity, and thus lends itself to very different logical traditions.

The precise conditions under which any given property is preserved through an inference is usually a matter of intense debate. The necessary and sufficient conditions for truth preservation are a prominent topic in discussions on logicality, as different notions of what it means for a consequence to be logical will lead to different requirements for the preservation of truth (BEALL; RESTALL; SAGI, 2019). In the context of proof or ground preservation, a precise account of what gives a valid inference its epistemic power has been singled out by Prawitz as the most fundamental problem of general proof theory (PRAWITZ, 2019a). The matter is far from settled in both cases, as characterizations of conditions for inferential property transmission are often quite diverse.

For our purposes, two properties are of special importance: *truth* and *proof*.

1.2.2 The concept of truth and its formal properties

Truth is traditionally taken to be one of the fundamental categories of the semantic analysis of logic. Its formal study was pioneered by Tarski, who gave rise to model theory by proposing mathematical definitions of satisfaction and truth that came to be widely accepted by logicians (TARSKI, 1944). Even before this formal characterization, it was already considered

[15] For a critical reading of this "transmission view", see (SCHROEDER-HEISTER, 2012).

one of the fundamental concepts of logic at least[16] since Aristotle (ROSS, 1949), together with modal concepts such as logical compatibility and inferential concepts such as syllogism. Many distinct philosophical theories aim to explain the nature of truth, and contemporary debates on this topic are very rich (HAACK, 1978, pgs. 86-134). In this work, however, we'll deal only with two kinds of theories: *coherence theories of truth* and *correspondence theories of truth*. Both usually agree that truth is a property of *propositions*[17], but disagree on what it means for a proposition to be true.

According to correspondence theories of truth, a proposition is true whenever it corresponds to reality, and false otherwise (DAVID, 2022). They adopt the definition of Aristotle, which withstood the test of time (ARISTOTLE, 2006, pg. 248):

> Truth is saying of what is the case that it is the case, or of what is not the case that it is not the case; falsity is saying of what is the case that it is not the case, or of what is not the case that it is the case.

Correspondence theories must rely both on some notion of *reality* and some notion of *correspondence* to define truth. The nature of reality and of this correspondence between reality and propositions must be fixed by the philosophical axioms of some particular theory, distinct axioms leading to distinct types of correspondence theories. Definitions of reality adopted by correspondence theorists are usually associated with metaphysical realism, and definitions of correspondence usually characterize it as either the result of mere linguistic conventions or as the result of some kind of structural isomorphism between propositions and reality (KIRKHAM, 1995, pg. 119), the most famous example of the latter being Wittgenstein's picture theory of language (WITTGENSTEIN, 2001).

According to coherence theories of truth, a proposition is true whenever it is coherent with some specified set of propositions (YOUNG, 2018). Depending on how strong the notion of coherence is, non-coherence of a proposition may or may not be sufficient for its falsity. While correspondence theories define truth in terms of correspondence and reality, coherence theories define it in terms of a notion of *coherence* and criteria for *specification* of the relevant set of propositions.

[16] It was also widely discussed by stoic logicians; see (MATES, 1961, pgs. 33–36) (BOBZIEN, 2003). In fact, many contemporary concepts were discussed by them, including truth-functional connectives and material implication (MATES, 1961, pg. 43–45). Susanne Bobzien argues that the many parallels between contemporary concepts and stoic notions are due to the fact that Frege, who is regarded by many as the founder of modern logic, essentially plagiarized the stoics (BOBZIEN, 2021).

[17] They agree that there are entities which we may call "propositions" that bear truth values, but not on what propositions themselves are. "Proposition" is used here as a synonym of "bearer of a truth value". The existence of some entity able to fulfill this role is taken for granted, but we remain agnostic with respect to theories determining what kind of entity this is. The longstanding debates concerning which linguistic or conceptual entities should be regarded as propositions (token statements, statements, conceptual content of statements, etc) are thus avoided. The reader is referred to (HAACK, 1978, pgs. 74-85) and (MCGRATH; FRANK, 2020) for comprehensive overviews of this subject.

Although coherence theories are not nearly as cohesive as correspondence theories, many of them are developed using the following rationale (PRIEST, 2005, pg. 50):

> Typically, those who endorsed the theory have held that it makes no sense to define truth in terms of some objective reality, independent of our cognitive functioning: there is no such thing, or if there is, we have no access to it. If we are to have any meaningful notion of truth, this can be defined only in terms of what we are justified in believing (maybe in the ideal limit). The criteria of coherence are therefore the criteria of justification.

Due to the fact that they do not rely on the existence of a subject-independent reality, coherence theories are usually adopted by metaphysical idealists. However, some particular brands of coherentism have also been advocated by non-idealists (NEURATH, 1983)(DA COSTA; BUENO; FRENCH, 2007).

Before we proceed, some remarks about relations between truth theories and our characterization of philosophical activity are in order. We have previously established that truth values are ascribed to propositions, regardless of what they are taken to be. We have also established that philosophical activity is reliant on the acceptance of philosophical axioms. We have not established, however, in what consists this acceptance, and it is certainly tempting to assume that acceptance of an axiom is acceptance of its evident truth. But this creates two problems: not only would the acceptance of a theory of truth already presuppose a notion of truth (applied to the axioms concerning its nature), but the distinction between correspondence and coherence theories would collapse.

To see why this is so, consider that any correspondence theory must axiomatically establish the constituents of the external reality to which propositions must correspond in order to be true. If that was not so, we would not be able to distinguish between reality and false perceptions of reality (or even between external reality and subject-dependent experiences). As such, what does and does not constitute reality must be fixed by axioms, which are also propositions. Since the truth conditions for propositions are now reliant on one or more propositions, by saying that a proposition corresponds to reality we are essentially saying that a proposition corresponds to something that another proposition establishes as being a part of reality, which is tantamount to saying that propositions are true whenever they are coherent with propositions which are taken to constitute reality.

Both problems are avoided if acceptance of an axiom is not conflated with acceptance of its truth. Correspondence and coherence theories are then distinguished by the fact that coherence is a relation between propositions and other propositions, whereas correspondence is a relation between propositions and an axiomatically (but not propositionally) established reality. It should be noted, however, that the desirability (or even possibility) of using truth as a primitive notion — whether for philosophy in general or for particular subjects — is still a matter of debate. A particular example is the controversy between Dummett and Davidson concerning language, in which Davidson's claim that the notion of meaning must be defined in terms of an irreducible notion of truth[18] is contested by Dummett, who argues that semantic theories (including truth-functional ones) must be preceded by a theory of meaning[19]. Even though we agree with Dummett that a sensible theory of sentential meaning should precede a sensible theory of sentential truth, we should not *a priori* preclude philosophers from claiming otherwise[20].

Both correspondence and coherence theories may be used for the purpose of establishing validity of inferences and arguments through property transmission. However, features commonly observed in coherence theories make the concept of truth very similar to other markedly epistemic notions, such as intuitionistic notions of *proof*. It is no coincidence that both intuitionism and coherence theories are usually defended by philosophers strongly opposed to metaphysical realism, as the notions of coherent truth and intuitionistic proof (or even intuitionistic mental construction) are independent of metaphysically realist notions such as that of a previously given external reality. But there may still be reasons to accept a notion of truth that is similar, albeit not identical to, an intuitionistic notion of proof — a pivotal point in some arguments of Prawitz and Dummett that will be examined later — and the best candidates for this would naturally be coherence theories.

To conclude our discussion of this topic, we will comment on three aspects of formalizations of concepts of truth that are important in the context of justification of inferences through property preservation. Those are usually seen all at once in correspondence theories, but some may be absent in coherence theories.

Truth is *monolithic*, in the sense that it is a single concept applied directly and uniformly to propositions. To say that there are multiple true propositions is to say that there are multiple

[18] "It is a misfortune that dust from futile and confused battles over these questions has prevented those with a theoretical interest in language - philosophers, logicians, psychologists, and linguists alike - from recognizing in the semantical concept of truth (under whatever name) the sophisticated and powerful foundation of a competent theory of meaning"(DAVIDSON, 1967).

[19] "This becomes evident if we imagine the theory stated by using, for the semantic values of sentences, not the familiar words 'true' and 'false', but some pair of hitherto unknown words. We should certainly then be under no impression that we had been provided with an adequate theory of meaning for the language. Even if we guessed that the two words denoted the two truth-values, we should not know which stood for the value true and which for the value false until we knew how the sentences were in practice used. It is what would have to be explained, concerning the newly introduced pair of terms, which we implicitly know concerning the terms 'true' and 'false', and which ought to be made explicit by any fully explanatory theory of meaning" (DUMMETT, 2006, pgs. 52-53).

[20] We must disclaim that here we are specifically considering the analysis of linguistic sentences (and thus restricting ourselves to philosophy of language), as the claim that meaning precedes truth is not necessarily incompatible with the view that acceptance of an axiom is acceptance of its truth. Remember that acceptance of an axiom must always be preceded by some primitive insight and the extraction of an axiom from it, so it would be perfectly reasonable to characterize meaning as one of the components of the extraction of axioms and truth as the property recognized during the acceptance of axioms. It would also be possible, however, to locate truth at a supra-axiomatic level, regardless of where meaning is located. In any case, the question concerning which notion should precede the other is relatively independent from the question concerning in what consists the acceptance of an axiom.

propositions to which the same semantic value (truth) is assigned, and truth-functional theories usually need to define only one notion of truth.

Truth is *total*, in the sense that it is usually defined for all appropriate propositions. Given any proposition, either truth applies to it (hence the proposition is true) or does not (hence it is false). A definite truth value must be ascribed to all propositions, and propositions are not allowed to be valueless[21].

Lastly, truth is *categorical*, in the sense that it applies unconditionally once defined. The truth of a proposition is contingent only on the obtainment of its truth conditions, regardless of its verifiability or our knowledge of it. After definitions of truth and falsity are supplied, the truth or falsity of every proposition immediately follows.

As will be shown, those aspects make truth-based formal theories very different from proof-based formal theories, which gives rise to important structural differences between model-theoretic and proof-theoretic semantics.

1.2.3 The concept of proof and its formal properties

The characterization of mathematical proofs as autonomous objects of study began with Hilbert's works on axiomatic systems (RATHJEN; SIEG, 2022), giving rise to what is nowadays known as "proof theory". Originally conceived in the context of a project aimed at providing formalist foundations for mathematics, which was famously frustrated by Gödel's incompleteness results (ZACH, 2005) (ZACH, 2023), proof theory was soon given new life by Gentzen's studies on logical deduction (GENTZEN, 1969, pg. 68-131) and Prawitz's furthering of Gentzen's natural deduction (PRAWITZ, 2006)[22][23].

The general concept of proof, with mathematical proof as one of its particular instances, has been studied for as long as the concept of truth. It has also been present in logic since the beginning. Not only did Aristotle study deductive proofs in his syllogistic doctrine (SMITH, 2022), the stoics also developed their own brand of syllogism through the notions of *indemonstrables* and *themata* (BOBZIEN, 1996), remarkably similar to contemporary proof-theoretic notions. Just as Tarski's truth definitions may be interpreted as an anachronistic formal counter-

[21] Although this is a core tenet of traditional semantics, it is rejected by some contemporary approaches, such as Belnap and Dunn's four-valued First-Degree Entailment (BELNAP, 2019)(DUNN, 1976).

[22] As Prof. Luiz Carlos Pereira puts it: Gentzen created natural deduction as most proof theorists know it, but Prawitz was responsible for providing it with citizenship status.

[23] Although Gentzen's approach became the dominant one, we should not forget that natural deduction was discovered independently by Jaśkowski (JAŚKOWSKI, 1934), in a formulation that was later given new notation by Fitch (FITCH, 1952) (PELLETIER; HAZEN, 2023). Fitch-Jaśkowski natural deduction is not only still used for pedagogical purposes in some books (MORTARI, 2001), but has also given its own independent contributions to logic and proof theory (INDRZEJCZAK, 1998).

part to Aristotle's definition of truth[24], Gentzen's sequent calculus and natural deduction may both be interpreted as formal counterparts of stoic deductive notions (BOBZIEN, 2019)(NASCIMENTO, 2023).

For the purposes of our study, a *proof* (or a *justification*) will be defined as *evidence* in favour of a given object or collection of objects having a certain quality[25]. Since assertions concerning the possession of a quality by an object may or may not be characterized as propositions, we use the word *statements* to refer to linguistic entities to which proofs apply (whatever their nature).

What counts as evidence is generally dependent on what kind of proof we are dealing with: mathematical proofs require mathematical evidence, whereas legal proofs require legal evidence. The typology of proofs is also dependent on the nature of the corresponding statements: mathematical statements require mathematical proofs through mathematical evidence, whereas legal statements require legal proofs through legal evidence. Obviously, what qualities a statement's object may have is also determined by the object's nature. Our concept of proof is thus reliant both on the concept of *statement* (which is reliant on the concepts of *object* and *quality*) and the concept of *evidence*, and this leads to some important differences with respect to definitions of truth.

Since statements, objects, qualities and evidence are not used in the previously given definitions of truth, both notions are taken to be independent. This should not be overlooked, as many philosophers argue in favour of there being a subordination between them. The literature contains both traditions in which the notion of truth is used to define proof and in which the notion of proof is used to define truth (MARTIN-LÖF, 1987).

This subject is especially contentious in debates concerning philosophy of logic. According to Dummett, it is incoherent to define a notion of truth that is independent of some notion of proof or justification, so the concept of proof must be prior to any concept of truth (DUMMETT, 1991). According to Tarski, however, proofs should be viewed as syntactic means to establish the truth of propositions, so the concept of truth must be prior to any concept of proof (TARSKI, 1969). The independence approach is also viable, and we could even argue that Plato's doctrine of knowledge as justified true belief (GETTIER, 1963)(PLATO, 1977) implicitly contains such an independence claim, since if truth presupposed justification, knowledge would simply be true belief, and if justification presupposed truth, knowledge would simply be

[24] This was explicitly stated by Tarski to be his intention (TARSKI, 1944).

[25] We once more define operational notions in order to avoid the many pitfalls of traditional discussions. It is not clear, for example, which treatment should be dispensed to non-existing objects, a subject dealt with extensively in Meinong's (sadly often misinterpreted and misjudged (SMITH, 1985)) Theory of Objects (MEINONG, 1904), Russell's famous response to it (RUSSELL, 1905) and Quine's equally famous later remarks (QUINE, 1948). Both "Bigfoot exists" and "Every positive even integer can be written as the sum of two primes" are considered statements subject to proof, but we will not delve into what it means for these objects to have the specified qualities. For a comprehensive overview of the subject, the reader is referred to (REICHER, 2022).

justified belief[26].

Another important aspect of our definition is that proof is treated as a strictly semantic notion. Proofs are semantic values preserved by inferences, not syntactic arguments. A vicious circle is created if, for example, one defines proofs by recourse to valid arguments and valid inferences as inferences that are proof preserving, since the notion of valid inference is already used in the definition of a valid argument (PRAWITZ, 2019b) (PRAWITZ, 2023a). In order to deal with this dilemma, one must choose between defining proof as a valid argument, thus refraining from using it as a semantic notion, or defining proof as a semantic notion, thus refraining from using it as a name for syntactic structures. Both solutions seem equally viable: Prawitz himself adopts the first and criticizes the second as being an inversion of the "natural conceptual order" (PRAWITZ, 2019b), whereas some approaches in proof-theoretic semantics defy this by defining proofs as the basic units of semantic analysis (SCHROEDER-HEISTER, 2022).

Although Prawitz argues that the best option would be to adopt a syntactic notion of proof, we claim that the semantic notion, aside from being very intuitive, provides a satisfying answer to the dilemma. The concept of proof is considered prior to the notion of argument, but this does not mean that arguments cannot provide us with proofs. Since proof-preserving arguments are syntactic structures enabling the production of a proof of the conclusion given proofs of the premises, the semantic content of an argument is a proof of the fact that from the justification of the premises follows the justification of the conclusion. Arguments are not proofs themselves, but the semantic content of an argument is always a proof. The existence of an argument is a sufficient, but not necessary, condition for the existence of a proof, and the vicious circle is avoided because proofs are no longer defined in terms of valid arguments. This also helps us in dealing with specific issues concerning mathematical proofs, as will be argued later.

Aside from having the intended practical effect of allowing us to claim the existence of a proof whenever we have an argument, an interesting consequence of this notion is that different proofs of the premises of an argument may produce different proofs of its conclusion. The existence of a procedure of proof conversion does not ensure by itself that for any proofs taken as input the same proof will be produced as an output. This naturally raises new questions concerning identity criteria between proofs, since it is not at all clear under which conditions different proofs of the premises produce the same proof of the conclusion. Prawitz's conjecture (or thesis) concerning proof identity (PRAWITZ, 1971) may still be used together with the semantic notion, since it's possible to claim that two arguments will have the same proof as their semantic content if and only if they are equivalent – but the question of identity criteria for

[26] For an argument in favour of precisely this kind of collapse, see (MARTIN-LÖF, 1998).

other kinds of proof remains open[27].

The uncontroversial epistemic nature of proof blurs the distinction between correspondence and coherence theories. Proofs are used to facilitate knowledge and knowledge can only be acquired by subjects, so it is usually agreed upon that proof is subject-dependent. Ontological notions such as truth usually require a great deal of metaphysical assumptions about the world in general, but epistemic notions require only assumptions concerning the knowing subject and the object of knowledge. One may still distinguish between correspondence and coherence theories, but it is not clear at all that this would be a helpful distinction. Correspondence theories could, for example, adopt epistemic notions according to which something is a proof only if it imparts cognitive changes (such as the formation of beliefs) in particular subjects, whereas coherence theories could adopt notions according to which proofhood is independent of concrete effects on particular subjects. Even it this case, however, it seems that disagreements would not be as deep as disagreements between defenders of correspondence and coherence theories of truth.

Just as in the case of truth, from the very concept of proof it follows that some behaviors are expected of any reasonable formal characterization. The differences between behaviors expected from formal notions of truth and proof are the very reason behind structural differences between model-theoretic proof-theoretic semantics.

Proof is *fragmentary*, in the sense that the general notion of proof characterizes a multitude of elements (distinct proofs) assigned in a non-homogeneous fashion to statements. Although "proof" may be treated as a single philosophical notion, a statement may have more than one proof, and it is even possible for some evidence to prove more than one statement [28].

Proof is *non-total*, in the sense that it is possible for a statement to have no proof value assigned to it. Given any object, there may or may not be evidence for it possessing or not possessing some particular quality. Since proofs are essentially dependent on some kind of human activity, any statement that never had any human action directed towards its epistemic constituents will have no evidence related to it, and thus no proofs[29].

Lastly, proof has both *categorical* and *hypothetical* aspects, in the sense that proofs may be conditional or unconditional even after a definition of what counts as evidence has been given. Unlike in the case of truth, assignment of proof values is not immediately provided

[27] For more on technical and philosophical aspects of proof identity and synonimity, the reader is referred to (WIDEBÄCK, 2001) (DOŠEN, 2003) (ALVES, 2019).

[28] The possibility of assigning more than one proof to the same statement is essentially the idea behind proof-based type-theories and "propositions-as-types" interpretations of logic, such as Martin-Löf's famous typed system (MARTIN-LÖF, 1985). It can also be argued that it is precisely this fragmentary nature that explains why some logics aimed at formalizing proof (such as intuitionistic and minimal logic) cannot be given finitely valued traditional semantics (GÖDEL, 1986, pg. 223).

[29] This naturally induces structural behavior commonly observed in logics rejecting the principle of bivalence, especially if proof of negation is interpreted as evidence of some object not possessing some quality (that is, as a refutation of the statement).

by proof conditions. The existence of particular proofs of statements must be supplied, so for a statement to have a proof some evidence must be produced. Evidences produced by a subject may be *conditional*, in the sense that they show that a statement is proved conditional on some other statement being proved, or *unconditional*, in the sense that they prove the statement directly. To give an example, some particular legal proof may be accepted unconditionally by a court, but it may also be accepted on the condition that some evidence be produced of the acquisition of the proof by lawful means.

Both categorical and hypothetical notions of proof can be encountered in the literature of proof-theoretic semantics. Prawitz's view of proof-theoretic validity in terms of closed arguments (PRAWITZ, 1971) (SCHROEDER-HEISTER, 2022) and Dummett's fundamental assumption (DUMMETT, 1991) are examples of semantic analysis conducted through categorical notions, while Popper's (POPPER, 2022) notion of abstract derivability and Sanz's (SANZ, 2022)(SANZ, 2019) notion of consequence relations for hypotheses may be cited as examples of analysis conducted through hypothetical notions. Although the dual nature of proof is often recognized, it is still common practice to define one aspect in such a way as to obtain the other as a byproduct. Categorical definitions are used more frequently due to established practices, but the desirability of this dogma has already been subjected to criticism (SCHROEDER-HEISTER, 2012).

1.3 **Special topics on truth and proof**

1.3.1 Intuitionistic truth and proof

The hegemonic truth-based approaches to logic were subjected to heavy criticism by mathematician and philosopher L. E. J. Brouwer in the 20th century. According to Brouwer, the only coherent way to characterize mathematics is as a languageless activity of mental construction by a creative subject (VAN DALEN, 1981). Mathematical objects are thus essentially subject-dependent, and any formal investigation in which no construction is effected is merely a linguistic effort devoid of any mathematical content. In particular, the traditional rule of *reductio ad absurdum* is no longer acceptable, as from a proof of the fact that the negation of a construction is inconsistent it does not follow that we have effected the construction itself. Mathematical theories of truth are also to be regarded as incoherent, inasmuch they characterize truth as not depending on any subject.

Brouwer's disruption led to the genesis of what is now known as the *intuitionistic* or *constructivist* conception of logic. Arend Heyting, one of his disciples, gave the first formalization of Brouwer's ideas in his *intuitionistic logic*, which is now taken as the main (but not only) formal representation of intuitionism (HEYTING, 1971). By then, intuitionism had become a highly heterogeneous school of thought (HESSELING, 2003). Intuitionists diverge on how notions of mental construction, proof or justification should be characterized, but they agree that subject-independent concepts of truth are not apt to justify mathematics or logic. Since distinct philosophical notions lead to distinct formalizations, the divergence also manifests itself at the level of formalism – a notable case being that of *miminal logic*, independently proposed by Ingebridt Johansson and Alexei Kolmogorov (JOHANSSON, 1937) (KOLMOGOROV, 2002) and recognized by Heyting himself as a viable formalization of intuitionistic thought (HEYTING, 1971, pg. 106).

The traditional enmity of intuitionists towards truth is justified only by the traditional association of truth with correspondence theories. Intuitionism is not *a priori* incompatible with theories of truth. In particular, since both reject subject-independent foundations, coherence theories are generally compatible with intuitionism. Some brands of intuitionism may regard truth as superfluous because any acceptable theory would collapse into notions such as proof or mental construction, but others consider the distinction desirable. It is in this vein that Dummett and Prawitz, for example, have argued in favour of the coexistence of non-collapsing intuitionistic notions of proof and truth. In the context of our framework, both Prawitz and Dummett would agree that propositions are also statements, and so that whatever is subject to truth values is also subject to proof. But their theories are very different: Dummett clearly adopts a coherence theory of truth, whereas Prawitz aims to provide a constructively acceptable correspondence theory.

In Dummett's conception, a statement can be asserted (or proved) whenever there is an actual construction of its object, and it is true whenever a procedure which leads to a construction of its object is available, even if the construction itself has not been obtained, and regardless of our knowledge of the fact that the procedure will lead to this construction (DUMMETT, 1998). Dummett prevents the collapse by establishing that some statements may be true because their construction will be an inevitable consequence of some available constructing procedure, even though they are not assertible due to our lack of knowledge that this construction will be produced. Dummett further argues that his theory should be adopted even outside of mathematics. This would allow interesting constructive interpretations of temporal phenomena, for example, but one still has to deal with problems faced by traditional conceptions – such as the Sea Battle problem, which may now be applied to statements concerning the past (PEREIRA, 2014).

The problem with this theory is that it makes the concept of truth very weak, in the sense that the distinction between truth and assertability becomes slim. This may be the reason that led Prawitz to ask whether Dummett was adopting the same conditions for truth and correct assertion, as reported in (DUMMETT, 1998, pgs. 122-123). It would be quite reasonable, even from an intuitionistic perspective, to adopt a stronger notion of proof which encompasses both what Dummett means by truth and what he means by assertability. This collapsed notion would still be intuitionistically acceptable, so the usefulness of such a distinction is not at all clear.

In Prawitz's conception, a statement can be asserted when there is a proof of it, and it is true when it is in principle provable (PRAWITZ, 1998). This is also the position of Martin-Löf (MARTIN-LÖF, 1987)(MARTIN-LÖF, 1998). The difference between truth and proof becomes the difference between actual and potential existence of proofs. Proofs are allowed to exist regardless of our knowledge of them, but a provable statement is considered to be merely true (thus being not assertible) until a proof is provided.

While this is a simple, intuitive and powerful distinction, Dummett argues (DUMMETT, 1998) that it inevitably takes us back to the realm of realist theories, and is thus not an intuitionistically acceptable theory of truth. This is so, he argues, because the concept of provability naturally allows one to define the category of unprovable statements by exclusion. Since every statement must either be provable or unprovable, the resulting concept leads to an intuitionistically unnaceptable principle of bivalence.

Prawitz responds to Dummett's accusation as follows (PRAWITZ, 1998, pg. 48):

> Although the idea of proofs existing independently of our hitting upon them certainly contains a flavour of realism, I do not think that it amounts to a full step to realism. I want to give two reasons for thinking so. Firstly, proofs as here understood are something that in principle can be known by us, and hence there is no talk about in principle unknowable proofs. Secondly, I do not see why the disjunction "either there exists a proof of A or there does not exist a proof of A" must be taken in a classical way. Although we think of the proofs as having some kind of existence even before we find them, an intuitionist may still maintain that to assert the disjunction that either there is or there is not a proof of A requires that we know how to find a verification either of the existence of a proof of A or of the non-existence of a proof of A. For an arbitrary A we do not know how to find such a verification, and we should then have no difficulty in resisting the thought that the disjunction in question is true.

Prawitz's first reason establishes that his concept of provability successfully prevents collapse with respect to notions of truth which admit in principle unknowable truths (although, of course, it does not immediately prevent collapse with realist notions admitting only knowable truths). However, Prawitz's second reason does not seem sufficient to repel Dummett's accusation. We can, in fact, interpret the statement "either there exists a proof of A or there does not exist a proof of A" constructively, so in order to assert it one would need to either produce a proof of A or show that A is impossible to prove. The problem is that Prawitz is discussing the disjunction's assertion conditions, not its truth conditions. In the presence of this distinction, it is not clear why by observing that the disjunction is not in general assertible we should have no difficulty resisting the thought that it is *true*, since it might be true *despite* not being assertible.

In fact, Prawitz's theory allows us to presuppose the existence of merely true statements, but not to assert it. In order to assert that there are true but non-assertible statements, we need to produce at least one statement that is known to be in principle provable without being assertible.

As Prawitz himself argues, we need to know how to verify a statement in order to assert it, so we need a verification showing that there is a proof we have not yet obtained. However, a constructive verification of the abstract existence of a proof would have to actually show this proof, so the statement itself would already be assertible[30]. A non-constructivist could claim that this is done by the rule of *reductio ad absurdum*, since by proving a contradiction from the assumption that a statement has no proof we guarantee that it has a not yet obtained proof, but this path is not available to the constructivist. A statement can thus only be said to be true if it is assertible, which beats the point of distinguishing between truth and proof conditions.

One might still try to avoid this by reading "verification" in a weakly constructive sense, so as to argue that possessing a verification of a statement does not entail possession of a proof of the statement, but only availability of a procedure which guarantees its production. But this is exactly Dummett's theory of truth, and we may still argue that the distinction is so weak that perhaps there is no use in it.

The existence of non-assertible truths is thus non-assertible, but perhaps one should be content with simply presupposing truths instead of asserting the existence of true statements. This is perfectly possible, albeit strongly realistic. The only difference between this position and the traditional ones is in what kind of realism we are dealing with. This debate is, of course, senseless to intuitionists that follow the tradition of rejecting the notion of truth altogether, but it leaves deviant ones in an awkward position. Intuitionistic assertion conditions (as defended by Dummett, Prawitz and Martin-Löf) are not taken to be a merely epistemic notion subservient to an ontological notion of truth, but a much stronger notion able to account for all meaningful human thinking. In order to obtain a truth theory capable of coexisting peacefully with an assertion theory in this sense, one must either weaken assertion conditions in order to create space for a coherentist truth[31] or renounce intuitionism altogether and adopt a concept of truth that transcends assertion.

At this point, it is necessary to remember that we have been discussing a strong, all-encompassing philosophical reading of intuitionism. It is still possible to adopt a strong but contained reading of intuitionism by arguing, for example, that truth is inadequate specifically

[30] This is recognized by Martin-Löf, who contends that in order to justifiedly assert the truth of a proposition one must have a proof of it (MARTIN-LÖF, 1998, pg. 112). He also accepts the existence of a "metaphysical truth" or "reality", which is taken as a given, but only after distinguishing it from truth as applied to propositions (MARTIN-LÖF, 1987, pgs. 418-420).

[31] Martin-Löf argues that intuitionism is actually a correspondence theory of truth (MARTIN-LÖF, 1998, pg. 112), with the novelty that propositions must correspond to proofs that exist in the world in order to be true. But this is, of course, very different from what is usually taken as the definition of a correspondence theory, so we prefer to characterize it as a coherence theory.

in the context of logic and mathematics[32]. It is also possible to adopt an ontological notion of truth together with an epistemic constructivism. Intuitionistic logic could then be viewed as a system concerned with how we are able to prove or refute the truth of a proposition, which is given beforehand and independently of any cognitive activity. This is a very sensible approach, and perhaps we could claim it is the prevailing one amongst students of intuitionistic logic that do not profess the intuitionistic faith.

1.3.2 Mathematical truth and proof

Correspondence theories have a hard time pinning down the truth conditions of mathematics. Mathematical propositions must correspond to states of affairs in order to be true, and most conceptions of what a state of affairs is lead to an undesirable reification of mathematical objects. If states of affairs are strictly empirical, mathematical truths must be empirical, and if mathematical truths are necessary, mathematical truths must be necessary empirical truths, subverting the whole idea of empiricism. On the other hand, if states of affairs are allowed to be merely conventional, truths may also be conventional; but conventions, unlike mathematical truths, may change at any time. It would be awkward to contend that Gödel's incompleteness results could be made false via agreement. It would also be awkward to claim that mathematical truths are conventional but immutable, since that would also subvert the very idea of what a convention is.

Problems of this kind are avoided altogether by those not particularly loyal to the material world. Coherence theories may satisfactorily explain mathematical truth in terms of coherence between mathematical propositions. Correspondence theories may satisfactorily explain mathematical truth if one fully embraces reification and concedes the existence of abstract ob-

[32] This is arguably the position of Heyting. In the opening dialogue of his introductory book on intuitionism (HEYTING, 1971), his representation of the intuitionistic position does not seem to dispute that there are non-mathematical subjects to which "traditional logic" correctly applies. We have underlined some parts of the relevant text which are taken to corroborate this reading:

> CLASS. Thank you. I bet you worked on that hobby of yours, rejection of the excluded middle, and the rest. I never understood why logic should be reliable everywhere else, but not in mathematics.
> INT. We have spoken about that subject before. The idea that for the description of some kinds of objects another logic may be more adequate than the customary one has sometimes been discussed. But it was Brouwer who first discovered an object which actually requires a different form of logic, namely the mental mathematical construction. The reason is that in mathematics from the very beginning we deal with the infinite, whereas ordinary logic is made for reasoning about finite collections.

jects, as do platonists[33] like Frege (FREGE, 1958). Intuitionists may also satisfactorily explain mathematical truth in terms of either correspondence or coherence with mental constructions, depending on how mental constructions themselves are characterized. This challenge poses itself only to those who wish to reconcile a material view of the world with the abstract and immutable nature of mathematics.

Intriguing answers are provided by what Benacerraf calls *combinatorial* theories of truth (BENACERRAF, 1973), which perhaps are the prevailing ones in the literature. According to combinatorialists, mathematical truths must be accounted for in terms of some relation between mathematical propositions and mathematical axioms. Arithmetical truth may be characterized, for example, as derivability from the Peano axioms, and by switching axioms one could also obtain other notions of mathematical truth. Collapse between notions of truth and proof are also avoided, since saying that something is derivable is not the same as saying that it has been derived. But, of course, combinatorial views preclude one from claiming the universal adequacy of formal theories such as Tarski's, since Gödel's incompleteness results would then decisively refute them.

There are many distinct flavours of combinatorialism, but perhaps their point could be put as as follows: mathematical truths are not conventions, but *consequences* of conventions. We are free to choose our axioms, but not what truths they beget. These consequences are not logical or mathematical in nature, since that would lead to an untenable, circular, all-encompassing conventionalism (WARREN, 2017)(QUINE, 1936), but we may characterize them empirically by reference to behavioral and linguistic phenomena. Mathematical truths are thus characterized as empirical consequences of linguistic conventions. To put it as an answer to the discovery versus invention debate: mathematical axioms are invented, but the truths they create must be discovered.

We deem this the most adequate answer to the initial dilemma, and thus the most convincing characterization of mathematical truth. It also has some peculiar features. Although clearly a correspondence theory, it is strikingly similar to Dummett's theory of truth. The main difference is that, in the combinatorial theory (as presented here), mathematical truths empirically follow from axioms fixed by convention, whereas in Dummett's theory they follow from constructions capable of showing those truths once carried out. In both cases some activity of a subject makes propositions true – regardless of whether the subject knows it – but in Dummett's case the subject must already be in a position to know those truths if he wishes. The combinatorial position may thus be viewed as a weakened, non-epistemic version of Dummett's position.

The characterization of mathematical proofs is considerably less controversial, mostly due to the fact that one does not need to advance particular ontological or metaphysical doc-

[33] Surprisingly, mathematical platonism may not be an accurate characterization of the views held by Plato (LINNEBO, 2023).

trines in order to talk about them. Regardless of how mathematical truths are conceived, a mathematical proof may be characterized as a conclusive proof of the truth of a mathematical proposition. Since the quality being proved is that of a proposition being true, the distinction between statements and propositions is no longer relevant.

The conclusive nature of mathematical evidence brings about some difficulties concerning validity of inferences via proof preservation. In Prawitz's view, conclusive proofs[34] are given by closed valid arguments, and closed valid arguments are unconditional arguments containing only valid inferences. But valid inferences are defined as inferences which preserve conclusive proofs, leading to interdependence at best and circularity at worst. Prawitz once argued that this interdependence was only apparent (PRAWITZ, 2019b), but later conceded that it was real (PRAWITZ, 2023a), although arguing that this is not necessarily a reason for us to abandon these definitions (PRAWITZ, 2023b).

It is quite natural to define a valid argument as an argument containing only valid inferences. It is also natural to claim that a valid argument yields some kind of proof, even in our weak sense. This is already sufficient for there to be some amount of interdependence, as valid inferences must preserve something which may have been given to us by a valid argument. But it is significantly more problematic to define – as Prawitz sometimes seems to – valid inferences as those capable of preserving *conclusive* proofs, as in this case the mutual dependece morphs into full circularity: an inference is valid just in case it preserves something obtainable only through the use of valid inferences.

Inferences capable of preserving proofs in general are also capable of preserving conclusive proofs in particular, hence this lax definition of validity is already sufficient for an adequate account of mathematical proofs. It also leads to a weaker form of interdependence, since validity of inferences is now defined prior to validity of arguments. This conceptual precedence cannot be achieved if validity of inferences is explained in terms of preservation of conclusive proofs; unlike the general notion of proof, the notion of conclusive proof already presupposes the notion of valid argument. As such, we conclude that by letting unconditional arguments be *sufficient*, but not *necessary* for the production of proofs we satisfactorily justify proof preservation without incurring full circularity.

This view can also be used to justify natural deduction and its procedures. Arguments always provide proofs, but not all proofs are provided by arguments – even though all conclusive proofs are provided by unconditional arguments. Just in case a proof is provided by an argument, it can be used in another argument to yield another proof; if we have an argumentative proof showing that from A follows B and another one showing that from B follows C, these can be put together to yield an argumentative proof showing that from A follows C. Likewise,

[34] Prawitz sometimes refers to conclusive proofs simply as "proofs", differing significantly from the terminology used here.

if we have a conclusive proof of A and an argumentative proof showing that from A follows B, we can meld those together to obtain a conclusive proof of B. This is, of course, nothing more than a semantic reading of the syntactic process of deduction composition.

We do not aim to solve all problems concerning mathematical truth and proof in this section, but we do claim to have shortly provided a sensible account of both. In any case, it seems clear that our definitions fulfill the important desiderata of defining mathematical proof and truth as particular cases of general notions instead of characterizing them as exceptional[35].

1.4 **Syntax and semantics**

1.4.1 Technical and conceptual semantics

The distinction between syntax and semantics was imported by logicians from the field of linguistics, but most of its original significance has been lost. Syntax was originally concerned with the structure of sentences, whereas semantics was concerned with their meaning. Systems of logic are traditionally defined through formal languages, so the distinction should still apply – and it does. When a logician says that a notion is syntactic or semantic, however, he expresses an idea barely related to the original distinction.

Given any syntactically specified structure (such as well-formed formulas), there are at least two senses in which another structure can be said to give a semantics for it: the *conceptual* sense and the *technical* sense. The conceptual sense is very close to the original linguistic distinction, but the technical sense is the prevailing one in mathematical logic. This is most likely explained by an undue association of formal semantics with model theory and Tarski's definition of truth. In fact, it is precisely due to the frequent conflation of formal semantics with model theory that Peter Schroeder-Heister coined the term "proof-theoretic semantics", rescuing the original meaning of semantics and challenging the traditional limitation of formal semantics to model-theoretic denotationalism (SCHROEDER-HEISTER, 2022)(SCHROEDER-HEISTER, 1991).

A structure is a semantics in the *conceptual sense* just in case it formalizes notions used to confer meaning to a syntactic structure. Tarski's truth theory is a semantics in this sense, since it gives a formal definition of truth that allows the meaning of syntactic sentences to be defined in terms of their truth value. A formalization of the notion of proof allowing proofs to be assigned to uninterpreted structures (say, arguments) would also be semantics in the conceptual sense. The only requirement is that a semantic notion must be defined, no constraints being imposed on structural aspects of the definition.

[35] Anti-exceptionalism is especially desirable if one wants to justify logic abductively (ERICKSON, 2021).

A structure is a semantics in the *technical sense* just in case it defines properties assigned to elements of an uninterpreted structure. Tarski's truth theory is also a semantics in this sense, as would be a proof-theoretic definition in which proof values are assigned to sentences. The technical sense is commonly used to differentiate between syntactic and semantic ways of specifying consequence relations: a relation is semantic if it is defined by recourse to properties assigned to classes of structures, and syntactic if it is defined by recourse to algorithmic processes. The only requirement is that the definition must have this structure, no constraints being imposed on the property being defined.

From the definitions it follows that the conceptual and technical senses are quite independent of each other. Definitions of formal semantics usually aim to be semantic in both senses, but there is no reason for us to exclude the possibility of semantics which are conceptually semantic but technically syntactic. On the other hand, it would be quite unreasonable to say that a definition in which an arbitrary property with no semantic bearing is assigned to sentences may also be called formal semantics. Precedence must be given to the conceptual sense, even though the technical sense also has its own worth.

This may be taken as an explanation of why many logicians are confused by the expression "proof-theoretic semantics": proof theory is mostly syntactic (in the technical sense), whereas model theory is mostly semantic (in the technical sense), whence it should make no (technical) sense to think of a proof-theoretic semantics. There is both a technical and a conceptual problem with this line of reasoning. For the conceptual problem, pointing out that formal semantics should be concerned primarily with semantics in the conceptual sense is already sufficient to show that not all semantics are model-theoretical. As for the technical problem, consider that the only requirement for a definition to be semantic is that it must define properties assigned to uninterpreted structures. There is no constraint on how such properties are to be defined, and they may as well be algorithmically specified. This is exactly what is done in some proof-theoretic semantics that define basic proofs by recourse to atomic (syntactic) derivability and then assign proof values to sentences based on the availability of atomic proofs.

Both senses are usually adopted in proof-theoretic semantics, as suggested by Schroeder-Heister's distinction between *semantics of proofs* and *semantics in terms of proofs* (SCHROEDER-HEISTER, 2022). When our definitions provide, for example, a conceptual proof semantics for syntactic derivations, we are dealing with a *semantics of derivations*; when they provide a technical proof semantics whose properties are given by syntactic derivations, we are dealing with a *semantics in terms of derivations*. Just as in the original distinction, the semantics of derivations may also be given in terms of derivations, in which case it would be a semantics in both senses.

1.4.2 Inferential and bilateral syntax and semantics

The inferential framework given in sections 2.1.3 and 2.1.4 may help us elucidate what is behind the (technical) distinction between syntax and semantics. A consequence relation may be defined procedurally, that is, through a procedure saying which consequences hold and which do not. One such procedure may be given by the use of valid inferences to produce valid arguments. This is clearly a syntactic definition, so it would be straightforward to use it in order to define natural deduction or sequent calculus. But, as shown in 2.1.3, validity in a model may also be defined through the use of (syntactic) arguments. Since the truth-functional (semantic) consequence relation is defined as holding just in case every model that makes the premises true also makes the conclusion true, we could inferentially define a truth-functional relation as holding whenever in all models on which there is an argument showing the truth of the premises there is also one showing the truth of the conclusion (or, more directly, whenever in all models there is a argument showing that from the truth of the premises follows the truth of the conclusion).

From this perspective, the technical distinction amounts to this:

Definition 1.4.1 Let I be a set of inferences and I^S a set of subsets of I. Define $\Gamma \vdash_D^K A$ as holding whenever there is a deduction D with premises Γ and conclusion A that uses only inferences contained in K. Then:

1. A syntactic relation $\Gamma \vdash_{syn} A$ holds if and only if there is a D such that $\Gamma \vdash_D^I A$;

2. A semantic relation $\Gamma \vdash_{sem} A$ holds iff for all $S \in I^S$ there is a D such that $\Gamma \vdash_D^S A$.

In other words, a syntactic definition quantifies existentially over deductions, whereas a semantic one quantifies universally (possibly over deductions in different systems). This approach has some interesting interactions with other frameworks that allow the comparison of syntactic and semantic notions, such as the bilateralist approach to consequence relations presented in (BLASIO; CALEIRO; MARCOS, 2019). In this approach, a S-consequence $\Gamma \rhd \Delta$ holds whenever the assertion of all formulas in Γ commits one to the assertion of all formulas in Δ, and a compatibility consequence $\Gamma \blacktriangleright \Delta$ holds whenever it is possible to simultaneously affirm all formulas in Γ and deny all formulas in Δ. S-consequences have a syntactic flavour to them, and compatibility consequences exhibits some behaviors one would expect from a semantic consequence relation.

Definition 1.4.2 A *S-consequence relation* is any relation $\rhd$ between sets of formulas and sets of formulas satisfying the following properties:

1. $\Gamma \rhd \Delta$ implies $\Gamma \cup \Gamma' \rhd \Delta \cup \Delta'$ (Monotonicity);

2. $\Gamma \cap \Delta \neq \emptyset$ implies $\Gamma \rhd \Delta$ (Reflexivity);

3. $\Sigma' \cup \Gamma \rhd \Delta \cup (\Sigma \setminus \Sigma')$ for every $\Sigma' \subseteq \Sigma$ implies $\Gamma \rhd \Delta$ (Cut).

Definition 1.4.3 A *compatibility relation* is any relation $\blacktriangleright$ between sets of formulas and sets of formulas satisfying the following properties[36]:

1. If $\Gamma \cup \Gamma \blacktriangleright \Delta \cup \Delta'$ then $\Gamma \blacktriangleright \Delta$ (Reducibility);

2. $\Gamma \blacktriangleright \Delta$ implies $\Gamma \cap \Delta \neq \emptyset$ (Disjointness);

3. If $\Gamma \blacktriangleright \Delta$, then there is some $\Sigma' \subseteq \Sigma$ such that $\Sigma' \cup \Gamma \blacktriangleright \Delta \cup (\Sigma \setminus \Sigma')$ (Expansion).

S-consequences can easily be used to provide syntactic procedures for the evaluation of consequence relations. For instance, classical sequent calculus can be defined by requiring $\rhd$ to satisfy rules for logical operations, as exemplified by the following:

1. $\Gamma \rhd_c \Delta \cup A$ implies $\Gamma \rhd_c \Delta \cup A \vee B$ (R$\vee$ rule);

2. $\Gamma \cup A \rhd_c \Delta$ and $\Gamma \cup B \rhd_c \Delta$ implies $\Gamma \cup A \vee B \rhd_c \Delta$ (L$\vee$ rule).

Validity of structural rules (Weakening, Contraction and Exchange) for the calculus follows immediately from the properties of S-consequences and the use of sets.

Compatibility relations may also be used to provide semantic procedures for consequence evaluation. Classical truth-functional semantics can be defined by requiring validity of all maximal consequences $\Gamma \blacktriangleright_c \Delta$ satisfying the usual semantic clauses:

1. $A \wedge B \in \Gamma$ if and only if $A \in \Gamma$ and $B \in \Gamma$;

2. $A \vee B \in \Gamma$ if and only if $A \in \Gamma$ or $B \in \Gamma$;

3. $A \rightarrow B \in \Gamma$ if and only if $A \notin \Gamma$ or $B \in \Gamma$;

4. $\neg A \in \Gamma$ if and only if $A \notin \Gamma$

5. For all A, $A \in \Delta$ if and only if $A \notin \Gamma$

Every formula is either asserted or denied according to the truth-functional clauses, so every maximal consequence $\Gamma \blacktriangleright_c \Delta$ represents a particular model. Non-maximal consequences represent the existence of at least one model making all formulas in Γ true and all in Δ false, since maximal consequences may be obtained via Expansion.

S-consequences naturally lend themselves to the direct specification of consequence relations, in contrast to compatibility relations' capability of allowing negative specification via absence of counterexamples:

[36] The names "Reducibility", "Disjointness" and "Expansion" are not used in the original work and were included only for pedagogical purposes.

Proposition 1.4.4 Classical syntactic consequence ($\vdash_c$), syntactic non-consequence ($\nvdash_c$), semantic consequence ($\vDash_c$) and semantic non-consequence ($\nvDash_c$) may be defined as follows:

1. $\Gamma \vdash_c \Delta$ if and only if $\Gamma \rhd_c \Delta$;
2. $\Gamma \nvdash_c A$ if and only if $\Gamma \rhd_c \Delta$ does not hold;
3. $\Gamma \vDash_c \Delta$ if and only if $\Gamma \blacktriangleright_c \Delta$ does not hold;
4. $\Gamma \nvDash_c \Delta$ if and only if $\Gamma \blacktriangleright_c \Delta$.

This is a very general framework, and by generalizing it further we could use $\rhd$ and $\blacktriangleright$ to define even more syntactic and semantic specification procedures. By allowing relations to be typed, defining 0-ary relations $\rhd^0$ as holding between sets of formulas and n-ary relations $\rhd^n$ (for $n > 1$) as holding between sets of $\rhd^{n-1}$ consequences, we could provide natural characterizations of higher-order natural deduction (SCHROEDER-HEISTER, 1984) and hypersequent calculus (CIABATTONI; RAMANAYAKE; WANSING, 2014). A similar generalization of $\blacktriangleright$ could be used to characterize Kripke semantics and define semantics for substructural and modal logics (RESTALL, 2018)(CHELLAS, 1980)[37].

In view of all of this, are we to conclude that $\rhd$ is an essentially syntactic relation and $\blacktriangleright$ a essentially semantic one? The answer is a resounding no, and we have many counterexamples to that. Syntactic characterizations of $\blacktriangleright$ are provided by complementary sentential logics (VARZI, 1990); in particular, classical complementary logic admits characterization via Hilbert systems, sequent calculi and proof nets (MORGAN, 1973)(PULCINI; CARNIELLI, 2015)(PULCINI; VARZI, 2023). Semantic uses of $\rhd$ are also frequent in proof-theoretic semantics, as exemplified by proof-theoretic validity (SCHROEDER-HEISTER, 2022) and others that rely on syntactic notions such as atomic derivability.

A more accurate reading of the bilateralist framework is one in which $\rhd$ is essentially a *proof-theoretic* relation, whereas $\blacktriangleright$ is essentially a *model-theoretic* one. The temptation to associate relations satisfying properties of $\blacktriangleright$ with semantics and those satisfying properties of $\rhd$ with syntax is yet another product of the conflation of formal semantics with model theory and syntactic procedures with proof theory. Proof-theoretic semantics can be obtained by quantifying universally over S-consequences, just as much as model-theoretic syntactic calculi can be obtained by quantifying existentially over compatibility consequences.

[37] Although the characterization of higher-order natural deduction seems to require all n-ary relations, traditional hypersequent calculi and Kripke semantics seem to require only arities 0 and 1. The sequents constituting a hypersequent would be 0-ary S-consequences, hypersequents themselves being 1-ary relations. Likewise, the Kripke worlds of a Kripke frame would be 0-ary compatibility relations, frames themselves being 1-ary relations. It would also be possible to allow both $\rhd^{n-1}$ relations and $\blacktriangleright^{n-1}$ relations to occur on the sets related by $\rhd^n$ and $\blacktriangleright^n$, leading to a generalized bilateral framework similar to the one presented in (BLASIO; CALEIRO; MARCOS, 2019). Further considerations on this subject are outside of the scope of this thesis, however.

1.4.3 Reasons for the adoption of proof-theoretic semantics

The aim of this thesis is to justify proof-theoretic semantics from a general point of view, not from the perspective of the philosophical doctrines often associated with it (such as intuitionism). Proof-theoretic semantics will have its development considerably hindered if it refuses to emancipate itself from those doctrines, as many logicians (this author included) do not find arguments in their favour very convincing – especially those to the effect that intuitionism should be adopted outside of logic and mathematics.

There are many technical and philosophical reasons for developing proof-theoretic semantics as a separate body of knowledge. Proof theory seems better suited for the representation of epistemic phenomena than model theory, which is arguably better at representing ontological phenomena. Model theory is currently used in many philosophical and technical projects for which it might not be the best available framework, either from a technical or conceptual standpoint.

A good reason for the study of proof-theoretic semantics is that fragmentary, non-total and hypothetical proof semantics are much more intuitive than their truth-functional (non-standard) counterparts. While the possibility of philosophical justification of truth-value gaps (that is, propositions with no truth value) is still controversial (SHAW, 2014), the non-total character of proofs makes it so that gaps are present from the start. The same happens with the multiplicity of semantic values: it is still a matter of debate whether many-valued logical notions such as that of truth degrees can be philosophically justified (GOTTWALD, 2022), but it is entirely reasonable to expect such fragmentary behavior from the notion of proof. Finally, the categorical nature of truth only allows one to entertain the notion of hypothetical judgements by introducing foreign epistemic or counterfactual elements (since hypothetical reasoning shows us what would be the case if some particular propositions were to be true, even though its truth or falsity is already determined), but some notions of proof prioritize the hypothetical over the categorical.

There is also an additional reason for the adoption of proof-preserving inferences instead of truth-preserving inferences in logic and mathematics: since in this context we are only interested in preserving conclusive proofs and conclusive proofs guarantee truth, proof-preserving inferences also preserve truth, whereas truth-preserving inferences do not necessarily preserve proof. Regardless of what definition of mathematical truth is adopted, the absolute epistemic grounds guaranteed by conclusive proofs have strong ontological bearings, but it is not generally the case that one can extract epistemic grounds from ontological grounds. A proposition for which we have absolute evidence is necessarily true, but propositions might be true despite our lack of evidence, so foundations built on the notion of proof are semantically more robust than those built on the notion of truth.

This list of justifications is not supposed to be exhaustive, especially since proof-theoretic semantics is a relatively new field of research. There is much yet to be considered, as its foundations are still being laid out. But there is already sufficient reason to believe that its development will yield many fruits, which is all that we need to start.

2 MATHEMATICAL GROUNDWORK

2.1 Basic definitions

2.1.1 Preliminary conventions

Whenever the number of syntactic objects used in a certain context becomes relevant (or when it improves readability), superscripts n will be used to distinguish between each element. Other syntactic markers are also used when convenient (e.g. S, S' and S'').

The distinction between a syntactic object and an occurrence of it will be taken for granted (e.g. there are two occurrences of the object "s" in the word "chess").

A sequence of objects will sometimes be said to contain a particular object whenever that object occurs in the sequence (in an abuse of set-theoretic terminology).

Whenever a relation is defined, a second relation is implicitly defined which holds if and only if the original relation does not hold. This new relation is denoted by putting a slash over the original relation (e. g. $\Gamma \nvdash A$ holds iff $\Gamma \vdash A$ does not hold).

The same syntactic object (e.g. $\vDash$) may be used to denote different notions whenever the notion being denoted is clear from the context.

Uppercase letters (A, B, C) are sometimes used to denote arbitrary objects of a kind specified by the definitions in which they appear.

The expression "iff" is sometimes used as an abbreviation for "if and only if".

Parentheses and commas will be dropped whenever no ambiguity ensues. They are also added in some occasions to improve readability.

2.1.2 On constants, variables and quantification

When introducing names for objects, logical languages usually distinguish between *constants* and *variables*. The object named by a constant is fixed, but we may change the denotation of variables. Variables are essential for the definition of quantification, since it only makes sense to refer to all possible denotations of a name when the denotation can actually be changed. There is no significant difference between both when quantification is not allowed, since using the fixed denotation of a constant or the denotation fixed by a particular interpretation function assigning objects to variables amounts to the same[38]. We will opt for the use of constants whenever there is a choice to be made, since this makes some comparisons between proof-theoretic and model-theoretic semantics smoother.

2.1.3 Languages

We use three kinds of basic syntactic symbols: constants, variables, and logical symbols[39]. Logical symbols (also called *connectives* or *operators*) are constants naming structurally fixed logical operations. As noted before, constants and variables are used to denote objects. We define $\perp$ as a predicate constant (instead of a logical symbol) in order to allow it to figure in second-order quantification.

Definition 2.1.1 A set is a *propositional language* if it contains the following elements:

1. An enumerable infinite number of predicate constants of arity 0. Notation: P, Q, R;
2. $\perp$, a special predicate constant of arity 0;
3. The logical symbols $\wedge$, $\vee$, $\rightarrow$.

Definition 2.1.2 A set is a *first-order language* if it contains the following elements:

1. An enumerable infinite number of predicate constants of arity n. Notation: P_n, Q_n, R_n;
2. $\perp$, a special predicate constant of arity 0;
3. An enumerable infinite number of individual variables. Notation: x, y, z;
4. Any number (possibly zero) of individual constants. Notation: a, b, c;
5. The logical symbols $\wedge$, $\vee$, $\rightarrow$, $\exists$ and $\forall$;

Definition 2.1.3 A set is a *second-order language* if it contains the following elements:

1. An enumerable infinite number of predicate variables of arity n. Notation: X_n, Y_n, Z_n;
2. $\perp$, a special predicate constant of arity 0;

[38] Some books don't even bother to specify if propositions and predicates are constants or variables, using names such as "propositional symbols" and "predicate symbols" until the need to make a distinction arises. For an example, see (VAN DALEN, 2013, pgs. 7, 56 and 145).

[39] Functions and the logical constant for equality are left out of definitions for the sake of simplicity. They may still be defined in terms of relations on the quantified calculi, so nothing is lost.

3. Any number (possibly zero) of predicate constants of arity n distinct from $\bot$;

4. An enumerable infinite number of individual variables;

5. Any number (possibly zero) of individual constants;

6. The logical symbols $\wedge$, $\vee$, $\rightarrow$, $\exists$ and $\forall$;

Definition 2.1.4 Every individual constant and individual variable of a language is a *first-order term* of that language. Every predicate constant and predicate variable of a language is a *second-order term* of that language.

The arity of predicate constants, variables and terms will sometimes be omitted when they are irrelevant or can be inferred from the context.

Definition 2.1.5 The set of *propositional formulas* of a language $\mathbb{L}$ is defined as follows:

1. Every 0-ary second-order term in $\mathbb{L}$ is a propositional formula of $\mathbb{L}$;

2. If A and B are propositional formulas of $\mathbb{L}$, then $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$ are propositional formulas of $\mathbb{L}$;

Definition 2.1.6 The set of *first-order formulas* of a language $\mathbb{L}$ is defined as follows:

1. If T_n is a second-order term in $\mathbb{L}$ and t^1 ... t^n is a sequence containing n first-order terms in $\mathbb{L}$, then $T_n t^1$ ... t^n is a first-order formula of $\mathbb{L}$.

2. If A and B are first-order formulas of $\mathbb{L}$, then $A \wedge B$, $A \vee B$ and $A \rightarrow B$ are first-order formulas of $\mathbb{L}$;

3. If A is a first-order formula of $\mathbb{L}$ and x is an individual variable in $\mathbb{L}$, then $\forall x A$ and $\exists x A$ are first-order formulas of $\mathbb{L}$;

Definition 2.1.7 The set of *second-order formulas* of a language $\mathbb{L}$ is defined as follows:

1. If T_n is a second-order term in $\mathbb{L}$ and t^1 ... t^n is a sequence containing n first-order terms in $\mathbb{L}$, then $T_n t^1$ ... t^n is a second-order formula of $\mathbb{L}$.

2. If A and B are second-order formulas of $\mathbb{L}$, then $A \wedge B$, $A \vee B$ and $A \rightarrow B$ are second-order formulas of $\mathbb{L}$;

3. If A is a second-order formula of $\mathbb{L}$ and x is an individual variable in $\mathbb{L}$, then $\forall x A$ and $\exists x A$ are second-order formulas of $\mathbb{L}$;

4. If A is a second-order formula of $\mathbb{L}$ and X is a predicate variable in $\mathbb{L}$, then $\forall X A$ and $\exists X A$ are second-order formulas of $\mathbb{L}$;

We speak simply of *formulas* whenever its specific type (propositional, first-order or second-order) can be inferred from the context.

Definition 2.1.8 The set of *atomic formulas* of $\mathbb{L}$ is defined as follows:

1. If T_n is a second-order term of $\mathbb{L}$ and $t^1 \ldots t^n$ is a sequence containing n first-order terms in $\mathbb{L}$, then $T_n t^1 \ldots t^n$ is a *atomic formula* of $\mathbb{L}$.

Definition 2.1.9 $\neg A$ is an abbreviation for $A \rightarrow \bot$.

We must also define substitution operations for formulas in order to deal with quantifiers in an orderly manner. There is more than one viable approach to this. We can, as does Prawitz (PRAWITZ, 2006, pg.13), adopt a syntactical distinction between variables (which always occur bound) and parameters (which always occur free). Another possibility is to use the approach of van Dalen (VAN DALEN, 2013, pg.61-62), in which substitution is defined only for unbound variables and for terms which are "free for" them in the formula (that is, terms which will not become bound after the substitution). Since constants will be used extensively later on and Prawitz's approach requires the use of an additional type of syntactic object, we adopt an approach similar to van Dalen's for the sake of linguistic simplicity.

Definition 2.1.10 A variable occurrence in a formula is *bound* in the following cases:

1. If A is an atomic formula, no variable occurrence in it is bound;

2. Every variable occurrence which is bound in A and in B is also bound in $A \wedge B$, $A \vee B$ and $A \rightarrow B$;

3. Every variable occurrence which is bound in A is also bound in $\exists x A$ and $\forall x A$. Additionally, every occurrence of x in $\exists x A$ and $\forall x A$ is bound.

4. Every variable occurrence which is bound in A is also bound in $\exists X A$ and $\forall X A$. Additionally, every occurrence of X in $\exists X A$ and $\forall X A$ is bound.

Definition 2.1.11 A variable occurrence is *free* in a formula if it is not bound.

Notice that the definition is given for occurrences of variables, not variables themselves. An occurrence of a variable is always either bound or free, but the same variable can occur both bound and free in the same formula (VAN DALEN, 2013, pg.60).

Definition 2.1.12 A *closed formula* (or *sentence*) is a formula with no free variable occurrences. Every formula which is not a closed formula is a *open formula*.

Definition 2.1.13 Let A be a open formula and $\langle X^1, \ldots X^n, x^{n+1}, \ldots, x^m \rangle$ a sequence containing all free predicate and individual variables of A by order of appearance (from left to right). Then $\forall X^1 \ldots \forall X^n \forall x^{n+1} \ldots \forall x^m (A)$ is the *universal closure* of A.

The notions of free and bound variable occurrences are defined in order to facilitate the definition of appropriate substitution operations. We give a naïve notion which is later refined by some constraints:

Definition 2.1.14 *Naïve substitution operations* for formulas A are defined as follows:

1. If t^1 and t^2 are first-order terms, then $A[t^1/t^2]^N$ is obtained by replacing every free occurrence of t^1 in A (if any) by an occurrence of t^2.

2. If T^1 and T^2 are second-order terms of the same arity, then $A[T^1/T^2]^N$ is obtained by replacing every free occurrence of T^1 in A (if any) by an occurrence of T^2;

Such substitutions are naïve because the terms t^1 and T^2 may become bound after the substitution (e.g. $(\forall x Pyx)[y/x]$, which yields $\forall x Pxx$).

In order to circumvent this, we define the following notions:

Definition 2.1.15 The term t^1 can be substituted by t^2 in in A if t^2 is not a bound variable in A. T^1 can be substituted by T^2 in A if T^2 if it not a bound variable in A.

Definition 2.1.16 *Substitution operations* (Notation: $A[t^1/t^2]$ and $A[T^1/T^2]$) are naïve substitution operations $A[t^1/t^2]^N$ and $A[T^1/T^2]^N$ such that t^1 can be substituted by t^2 and T^1 by T^2 in A.

Whenever $A[t^1/t^2]$ and $A[T^1/T^2]$ are used, we assume that the terms t^2 and T^2 are being chosen in such a way that there is a substitution operation available.

Definition 2.1.17 *Subformulas* of formulas are defined as follows:

1. Every formula is a subformula of itself;

2. A and B are subformulas of $A \wedge B$, $A \vee B$ and $A \to B$;

3. If t is a first-order term and T is a second-order terms, then $A[x/t]$ is a subformula of $\forall x A$ and $\exists x A$, and $A[X/T]$ is a subformulas of $\forall X A$ and $\exists X A$.

Definition 2.1.18 The *degree* or *complexity* of a formula is the number of logical operators occuring in it.

Degrees are defined for the sake of inductive proofs, so all inductive steps are inductions on the degree of formulas unless otherwise specified.

Before proceeding to the next section, we introduce new syntactic objects called *formula variables*. Those will be used to define schemas, which are placeholders for formulas. The notation (ϕ, ψ, σ) will be employed.

Definition 2.1.19 A *formula schema* is defined as follows:

1. If ϕ is a formula variable, then ϕ is a formula schema;

2. If ϕ and ψ are formula schemas, then $\phi \wedge \psi$, $\phi \vee \psi$, and $\phi \to \psi$ are formula schemas. If x is a individual variable and X a predicate variable, then $\exists x\phi$, $\forall x\phi$, $\exists X\phi$ and $\forall X\phi$ are also formula schemas.

Definition 2.1.20 Let $\{\phi^1, \ldots, \phi^n\}$ be the set of all formula variables occurring on a formula schema and $\{A^1, \ldots, A^n\}$ be a set of formulas. An *instance* of that formula schema is the result of replacing all occurrences of variables ϕ^m in it by A^m $(1 \leq m \leq n)$.

Since our main interest lies in the relation between formulas, formula schemas will be used as representatives of their instances. Notions involving validity and deduction are defined for formulas, but schemas allow us to deal with them in a uniform manner.

2.2 **Natural deduction**

2.2.1 Rule schemas, rules and deductions

We will now give definitions for Gentzen-style natural deduction (GENTZEN, 1969, pg. 68-80), our main deductive framework. Our definitions are similar to those used by van Dalen (VAN DALEN, 2013), but with some elements taken from Prawitz (PRAWITZ, 2006). We define two kinds of objects: *natural deduction rules* (or inferences), in which formulas are used as premises to conclude another formula, and *natural deduction rule schemas*, in which formula schemas and formulas are used as premises to conclude a formula schema or formula. Just as in the case of formulas and formula schemas, a rule schema is meant to be read as a placeholder for particular rules.

Definition 2.2.1 Let $\langle A^1, \ldots, A^n\rangle$ be a sequence of n formulas, $\langle \Delta^1, \ldots, \Delta^n\rangle$ a sequence of n sets of formulas and B a formula. Then:

$$\dfrac{\begin{matrix}[\Delta^1] \\ \vdots \\ A^1\end{matrix} \quad \ldots \quad \begin{matrix}[\Delta^n] \\ \vdots \\ A^n\end{matrix}}{B}$$

Is a *natural deduction rule*. We represent this horizontally by writing $[\Delta^1 \Rightarrow A^1, \ldots, \Delta^n \Rightarrow A^n/\ B]$, and $\Delta^m \Rightarrow A^m$ may be simplified to A^m when Δ^m is empty.

Definition 2.2.2 Let $\langle\phi^1_*,\ \dots, \phi^n_*\rangle$ be a sequence of schemas and/or formulas, $\langle\Theta^1_*,\ \dots, \Theta^n_*\rangle$ a sequence of n sets containing schemas and/or formulas and ψ_* a formula schema or formula. Then:

$$\dfrac{\begin{matrix}[\Theta^1_*] & & [\Theta^n_*] \\ \vdots & & \vdots \\ \phi^1_* & \dots & \phi^n_*\end{matrix}}{\psi_*}$$

Is a *natural deduction rule schema*. We represent this horizontally by $[\Theta^1_* \Rightarrow \phi^1_*,\ \dots, \Theta^n_* \Rightarrow \phi^n_* /^R\ \psi_*]$, and $\Theta^m \Rightarrow \phi^m_*$ may be simplified to ϕ^n_* when Θ^m is empty.

In both cases, the formulas/formula schemas with superscripts 1 through n are called the *premises* of the rule/rule schema, and the formulas/formula schemas without superscripts its *conclusion*. The formulas in the sets Δ^1 through Δ^n and formulas/formula schemas in Θ^1 through Θ^n are *discharged* by the rule/rule schema. Notice that the premises are not necessarily distinct from each other, from the conclusion or from formulas/formula schemas in the discharged sets.

Definition 2.2.3 Let $\{\phi^1,\ \dots, \phi^n\}$ be the set of all formula variables occurring in some formula schema in a natural deduction rule and $\{A^1,\ \dots, A^n\}$ any set of formulas. A *uniform substitution* for that rule is the result of replacing all occurrences of each formula variable ϕ^m in every formula schema of the rule by the formula A^m ($1 \leq m \leq n$).

Definition 2.2.4 A natural deduction rule is an *instance* of a natural deduction rule schema iff the rule can be obtained from the rule schema via uniform substitution.

Natural deduction rule schemas are allowed to use both formulas and schemas. Formula occurrences are preserved by uniform substitution, so whenever a formula occurs in a rule schema its position is fixed in all of its instances.

Definition 2.2.5 A *deduction* (also called *derivation*) of a formula A depending on a set Γ of formulas, denoted by $\begin{matrix}\Gamma \\ \Pi \\ A\end{matrix}$, is defined as follows:

1. If A is a formula, A itself is a deduction of A depending on $\Gamma \cup \{A\}$, for any Γ;

2. If $\begin{matrix}\Gamma^m \\ \Pi^m \\ A^m\end{matrix}$ is a deduction of A^m depending on Γ^m for $(1 \leq m \leq n)$ and $\dfrac{\begin{matrix}[\Delta^1] & & [\Delta^n] \\ \vdots & & \vdots \\ A^1 & \dots & A^n\end{matrix}}{B}$ is a natural deduction rule, then $\dfrac{\begin{matrix}\Gamma^1, [\Delta^1] & & \Gamma^n, [\Delta^n] \\ \Pi^1 & & \Pi^n \\ A^1 & \dots & A^n\end{matrix}}{B}$ is a deduction of B depending on $\bigcup\limits_{1 \leq k \leq n} \Gamma^k - \Delta^k$.

Definition 2.2.6 A formula A is a *assumption* of a deduction of B if and only if it the deduction of B depends on Γ and $A \in \Gamma$. B may also be said to *depend* on A.

Definition 2.2.7 A deduction is *closed* if it depends on the empty set.

Definition 2.2.8 A *restriction* on a natural deduction rule schema is any condition imposed on its formula schemas or uniform substitutions.

Definition 2.2.9 A deduction is *valid* in a given logic iff all formulas occurring in it are of the appropriate language, all natural deduction rules in it are instances of a natural deduction rule schema considered valid by that logic and all restrictions imposed on the rules schemas are satisfied by its instances.

2.2.2 Propositional logic

We are now equipped to define rule schemas used in natural deduction systems of propositional, first-order and second-order minimal, intuitionistic and classical logic. Minimal propositional logic is used as a starting point; intuitionistic and classical propositional logic are obtained by adding new rule schemas to it. In the same fashion, the first-order version of a logic is obtained by using a first-order language and adding rule schemas for the first-order quantifiers to its propositional system, whereas second-order versions are obtained by using a second-order language and adding rule schemas for second-order quantifiers to the first-order systems.

Definition 2.2.10 A natural deduction system for *minimal propositional logic* is obtained by using a propositional language and regarding the following schemas as valid:

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\, I\wedge \qquad \frac{\phi \wedge \psi}{\phi}\, E\wedge^1 \qquad \frac{\phi \wedge \psi}{\psi}\, E\wedge^2$$

$$\frac{\phi}{\phi \vee \psi}\, I\vee^1 \qquad \frac{\psi}{\phi \vee \psi}\, I\vee^2 \qquad \frac{\phi \vee \psi \qquad \begin{array}{c}[\phi]\\ \vdots \\ \sigma\end{array} \qquad \begin{array}{c}[\psi]\\ \vdots \\ \sigma\end{array}}{\sigma}\, E\vee$$

$$\frac{\begin{array}{c}[\phi]\\ \vdots \\ \psi\end{array}}{\phi \to \psi}\, I\to \qquad \frac{\phi \qquad \phi \to \psi}{\psi}\, E\to$$

Definition 2.2.11 A natural deduction system for *intuitionistic propositional logic* is obtained by using a propositional language and regarding all rule schemas of minimal propositional logic as valid, plus the rule of *ex falso quodlibet*:

$$\frac{\bot}{\phi}\ \text{EFQ}$$

This schema is different from the minimal ones due to its use of a formula; the occurrence of $\bot$ in the premise is fixed in all instances of EFQ.

Definition 2.2.12 A natural deduction system for *classical propositional logic* is obtained by using a propositional language and regarding all rules of intuitionistic propositional logic as valid[40], plus the rule schema of *reductio ad absurdum*:

$$\frac{\begin{array}{c}[\neg\phi]\\ \vdots\\ \bot\end{array}}{\phi}\ \text{RAA}$$

Propositional syntactic consequence is defined for the three logics as follows:

Definition 2.2.13 Let $\mathbb{L}$ be a propositional language. Let Γ be a set of propositional formulas and A a propositional formula of $\mathbb{L}$. The relations $\vdash^0_m$, $\vdash^0_i$ and $\vdash^0_c$ hold between them if and only if there is a deduction with premises Γ and conclusion A using only the valid rules of minimal, intuitionistic and classical propositional logic, respectively.

2.2.3 First-order logic

We use $\phi[x/t]$ as shorthand for a restriction: if ϕ and $\phi[x/t]$ occur in the same rule, then every uniform substitution which replaces ϕ by A must also replace $\phi[x/t]$ by $A[x/t]$. Restrictions are imposed on instances of rules schemas, not the schemas themselves.

Definition 2.2.14 Natural deduction systems for *minimal, intuitionistic and classical first-order logics* are obtained by using a first-order language and regarding the rule schemas of minimal, intuitionistic and classical propositional logic as valid (respectively), plus the following first-order quantification rule schemas:

[40] Or even of minimal logic, as EFQ is a particular instance of RAA.

$$\frac{\phi}{\forall x(\phi)}\, I\forall^1 \qquad \frac{\forall x(\phi)}{\phi[x/t]}\, E\forall^1 \qquad \frac{\phi[x/t]}{\exists x(\phi)}\, I\exists^1 \qquad \frac{\exists x(\phi) \qquad \begin{array}{c}[\phi[x/t]] \\ \vdots \\ \psi\end{array}}{\psi}\, E\exists^1$$

Restriction on $I\forall^1$: x does not have free occurrences on formulas on which ϕ depends;

Restriction on $E\exists^1$: t does not have free occurrences on ψ, ϕ or any formula on which the upper occurrence of ψ depends other than $\phi[x/t]$;

First-order syntactic consequence is defined as follows:

Definition 2.2.15 Let $\mathbb{L}$ be a first-order language. Let Γ be a set of first-order formulas and A a first-order formula of $\mathbb{L}$. The relations $\vdash^1_m$, $\vdash^1_i$ and $\vdash^1_c$ hold between them if and only if there is a deduction with premises Γ and conclusion A using only the valid rules of minimal, intuitionistic and classical first-order logic, respectively.

2.3 Model-theoretic Semantics

2.3.1 Propositional logic

In this section we present standard Kripke semantics (VAN DALEN, 2013, pgs. 164-165)(COLACITO; JONGH; VARGAS, 2017) for propositional and first-order minimal, intuitionistic and classical logic. Second-order logic is left for later. We start by dealing with minimal logic and later use it to define intuitionistic and classical logic.

Definition 2.3.1 Let $\mathbb{L}$ be a propositional language. A *minimal propositional model* K for it is any sequence $\langle W, \leq, v\rangle$ such that:

1. W is a non-empty set of objects k;

2. $\leq$ is a partial order on the elements of W;

3. v is a *valuation function*, assigning either the value T (true) or the value $\emptyset$ (non-true) to every ordered pair (A, k) comprised of *atomic sentences* A (cf. Definitions 2.1.5 and 2.1.8) and objects k of W. Valuation functions must satisfy the requirement that if $k \leq k'$ and $v(A, k) = T$ then $v(A, k') = T$.

W corresponds to the usual set of "worlds" or "nodes"; $\leq$ is the usual accessibility relation. The semantic value "non-true" is used so as to avoid confusion with standard (classical) semantic definitions of falsity. The function v establishes what atoms are true in each world k.

The condition imposed on v is called the *heredity* or *monotonicity condition*, and it guarantees that whenever the truth of an atom has been established in a world it must also be established in all accessible worlds. Although the condition is imposed on atoms, it is possible to prove that it spreads over to all standard operators[41].

Once models have been defined, we may define semantic consequence for minimal propositional logic as follows, fixing $\Longleftrightarrow$ as a synonym of "iff":

Definition 2.3.2 Let $\mathbb{L}$ be a language. Given a minimal propositional model K, the relations $\vDash_k^K$ and $\vDash^K$ are defined for the sentences of the language as follows:

1. $\vDash_k^K A \Longleftrightarrow v(A,k) = T$, for atomic sentences A;
2. $\vDash_k^K A \wedge B \Longleftrightarrow \vDash_k^K A$ and $\vDash_k^K B$;
3. $\vDash_k^K A \vee B \Longleftrightarrow \vDash_k^K A$ or $\vDash_k^K B$;
4. $\vDash_k^K A \rightarrow B \Longleftrightarrow A \vDash_k^K B$;
5. $\vDash_k^K \Gamma \Longleftrightarrow \{\vDash_k^K A_i \mid A_i \in \Gamma\}$, where Γ is a set of propositional sentences;
6. $\Gamma \vDash_k^K A \Longleftrightarrow \forall k'(k \leq k') : \vDash_{k'}^K \Gamma$ implies $\vDash_{k'}^K A$;
7. $\Gamma \vDash^K A \Longleftrightarrow \forall k(k \in W) : \Gamma \vDash_k^K A$.

Definition 2.3.3 A minimal propositional model K is also an *intuitionistic propositional model* just in case $v(\perp, k) = \emptyset$ for all $k \in W$.

Definition 2.3.4 An intuitionistic propositional model K is also a *classical propositional model* just in case $\nvDash_k^K A$ and $k \leq k'$ implies $\nvDash_{k'}^K A$.

To see why classical propositional models are enough to give us classical logic, notice that the new condition makes quantification over accessible worlds redundant, since either $\vDash_k^K A$ and thus $\vDash_{k'}^K A$ for every $k \leq k'$ (heredity condition) or $\nvDash_k^K A$ and thus $\nvDash_{k'}^K A$ for every $k \leq k'$ (classical condition). In particular, clause 4 of Definition 2.3.2 boils down to ($\vDash_k^K A \rightarrow B \Longleftrightarrow \nvDash_k^K A$ or $\vDash_k^K B$), since for $A \vDash_k^K B$ to be satisfied either $\nvDash_k^K A$ and the relation is satisfied vacuously or $\vDash_k^K A$ and then $\vDash_k^K B$. Since $\neg A$ is a abbreviation of $A \rightarrow \perp$, in every k of a classical model we also have that either $\vDash_k^K A$ holds or $\nvDash_{k'}^K A$ holds for every $k \leq k'$, which vacuously satisfy clause 4 of Definition 2.3.2 and yields $\vDash_k^K \neg A$, from which we conclude $\vDash_k^K A \vee \neg A$. Definition 2.3.4 makes it so that accessible worlds become completely irrelevant, which leads to a collapse into the traditional models of classical logic. However, it should be noted that it is enough to require preservation of the value $\emptyset$ assigned to atoms to induce classical

[41] For a quick proof, see the footnotes in (PRIEST, 2008, pgs. 105-106).

behavior in propositional models, but some problems arise in the first-order and second-order cases.

Propositional semantic consequence is defined as follows:

Definition 2.3.5 Let $\mathbb{L}$ be a propositional language. Let Γ be a set of propositional formulas and A a propositional formula of $\mathbb{L}$. The relations $\vDash^0_m$, $\vDash^0_i$ and $\vDash^0_c$ hold between them if and only if $\Gamma \vDash^K A$ holds for every minimal, intuitionistic and classical propositional model K, respectively.

2.3.2 First-order logic

We need to introduce the notions of domain and interpretation before defining first-order minimal models. It is also necessary to choose how to deal with formulas containing free variables, since valuations are defined only for sentences. Before going to the definitions, we briefly fix the convention that, whenever a function f is defined, $f(s)$ will be used to denote the value of the function when applied to the argument s.

Definition 2.3.6 A *domain* D is any non-empty set of objects.

Definition 2.3.7 A *domain assignment function* α for a set W is a function assigning a domain $\alpha(k)$ to every $k \in W$ and $\bigcup_{k \in W} \alpha(k)$ to W itself.

Definition 2.3.8 Let $\mathbb{L}$ be a language, W a set of objects k and α a domain assignment function for W. An *interpretation function* β is a function assigning to every constant of the language an element of $\alpha(W)$ and to every pair (P_n, k) of n-ary predicate constants of $\mathbb{L}$ (for $n \geq 1$) and elements k of W an n-ary relation $(a^1, \, \dots \, , a^n)$ on elements of $\alpha(k)$.

Definition 2.3.9 Let α be a domain assignment function for W. The language $\mathbb{L}(\alpha(W))$ is obtained by adding to $\mathbb{L}$ one distinct individual constant for each element of $\alpha(W)$.

First-order models are defined only for languages with at least one constant for every object of its domains. Instead of imposing a restriction on languages, we may start with any language and, given a model, enrich it with enough constants to name all objects of all domains. The function α will be used to specify the domains $\alpha(k)$ of every k and the set $\alpha(W)$ of all objects occurring in some domain, so we may use β to assign the same name to an object even if it occurs in different domains. β will also used to induce a particular behavior on the function v.

First-order minimal models are defined as follows:

Definition 2.3.10 Let $\mathbb{L}$ be a first-order language. A *first-order minimal model* K for the extended language $\mathbb{L}(\alpha(W))$ is a sequence $\langle W, \leq, \alpha, \beta, v \rangle$ such that:

1. W and $\leq$ are as in Definition 2.3.1;

2. α is a domain assignment function for W, satisfying the conditions that $k \leq k'$ implies $\alpha(k) \subseteq \alpha(k')$;

3. β is an interpretation function for $\mathbb{L}$, W and α, satisfying the conditions that $k \leq k'$ implies $\beta(P_n, k) \subseteq \beta(P_n, k')$ and, for every element e of $\alpha(W)$, there is a constant $a \in \mathbb{L}(\alpha(W))$ such that $\beta(a) = e$;

4. *v* is as in Definition 2.3.2 for 0-ary predicate constants. For predicates of greater arity, $v(P_n(a^1, \ldots, a^n), k) = T$ iff $\langle \beta(a^1), \ldots, \beta(a^n) \rangle \in \beta(P_n, k)$[42].

Definition 2.3.11 Let $\mathbb{L}$ be a first-order language and K a first-order minimal model. The relations $\vDash_k^K$ and $\vDash^K$ are defined for the sentences of $\mathbb{L}(\alpha(W))$ as follows:

1. Clauses 1 through 9 are the same as in Definition 2.3.2;

10. $\vDash_k^K \forall x(A) \iff \forall k'(k \leq k') : \vDash_{k'}^K A[x/a]$, for all a such that $\beta(a) \in \alpha(k)$;

11. $\vDash_k^K \exists x(A) \iff \vDash_k^K A[x/a]$, for some a such that $\beta(a) \in \alpha(k)$;

The clauses are defined only for sentences, but we may extend them to open formulas as follows:

Definition 2.3.12 If A is an open formula and A^* its universal closure, $\vDash_k^K A \iff \vDash_k^K A^*$.

Classical and intuitionistic first-order models are defined by imposing the same propositional conditions (Definitions 2.3.3 and 2.3.4) on minimal first-order models.

As mentioned before, Definition 2.3.4 is necessary to induce classical behavior in predicate logics, while in the propositional case it is enough to require preservation of the values of atoms. To see why this is so, consider a model K with a k such that $\alpha(k) = \{a\}$, a k' such that $k \leq k'$ with $\alpha(k') = \{a, b\}$ and a k'' such that $k \leq k''$ with $\alpha(k'') = \{a, c\}$. Suppose now that, for all k of K, $\vDash_k^K Pa$ and $\vDash_k^K Pb$, but $\nvDash_k^K Pc$. It is easy to see that, provided there are no other worlds accessible from k, k' and k'', we have $\vDash_{k'}^K \forall x Px$, $\nvDash_{k''}^K \forall x Px$, and $\nvDash_k^K \forall x Px$. But, since $k \leq k'$, we also have $\nvDash_k^K \neg \forall x Px$, and so $\nvDash_k^K (\forall x Px) \vee \neg(\forall x Px)$, which takes us away from classical logic.

Finally, we may define first-order semantic consequence as follows:

Definition 2.3.13 Let $\mathbb{L}$ be a first-order language. Let Γ be a set of first-order formulas and A a first-order formula of $\mathbb{L}$. The relations $\vDash_m^1$, $\vDash_i^1$ and $\vDash_c^1$ hold between them if and only if $\Gamma \vDash^K A$ holds for every minimal, intuitionistic and classical first-order model K, respectively.

[42] It is not necessary to impose the heredity condition on predicates with arity greater than 0, as the property follows from the definition of β.

2.4 **Second-order logic**

2.4.1 On the nature of second-order logic

The name “second-order logic” is used ambiguously in the literature. Although sometimes used to refer to standard models of second-order logic, for which incompleteness results infamously hold (EBBINGHAUS; FLUM; THOMAS, 1984, pgs. 162-165), it is also used to refer to a calculus created by Church (CHURCH, 1956, pgs. 295-302), subsequently used as the main definition of second-order logic in many studies (PRAWITZ, 2006, pgs. 63-73) (VAN DALEN, 2013, pgs. 145-152). It may also reasonably be used for Henkin’s F^* calculus (HENKIN, 1953), a proper subsystem of Church’s calculus.

For convenience, we may refer to the standard models as *strong second-order logic*, to Church’s calculus as *intermediate second-order logic* and to Henkin’s F^* calculus as *weak second-order logic*. Our main source on this subject will be the study conducted by Manzano (MANZANO, 1996), in which Henkin’s definition corresponds to the calculus C_2^- and Church’s definition corresponds to C_2. Due to the incompleteness results, no calculus can be given for strong second-order logic[43].

The calculus we’ll use corresponds to *weak second-order logic*, and thus to Manzano’s C_2^-. Our semantic discussions also focus mostly on the weak logic, but we’ll always comment on ways of obtaining semantics for the intermediate and strong logics. To justify this choice, we will now present three reasons for the use of the weaker version.

The first one is *syntactic simplicity*. Quantification works in the same way for individuals and predicates in the weak logic. The weak second-order calculus is a straightforward extension of the first-order one, even though it does not have properties some would expect of second-order logics – such as derivability of all instances of the comprehension schema, according to which $\exists X \forall x^1 \ ... \ \forall x^n (X x^1 \ ... \ x^n \leftrightarrow A)$, provided X does not have free occurrences on A. Intermediate second-order logic, on the other hand, requires either that one (i) Substantially change how substitution works in second-order quantification (VAN DALEN, 2013, pg. 147); (ii) Add rules allowing one to prove all instances of the comprehension schema (MANZANO, 1996, pg. 79), or (iii) Define an independent λ operator, which interacts with the quantifier rules and allows one to prove all instances of the comprehension schema (PRAWITZ, 2006, pg. 66). In all cases, the behavior of second-order substitution becomes quite unnatural, and the desired properties are induced either through direct stipulation, through indirect interference of the λ operator, or through effects of the modified definition of substitution on second-order quantifications.

[43] For a interesting but brief discussion on the differences between the three definitions of second-order logic, the reader is referred to (MANZANO, 1996, pgs. 73-75).

The second is *semantic generality*. It is usually the case that semantic definitions sound and complete for C_2^- can be extended to C_2 by simply imposing the requirement that all models must satisfy all instances of the comprehension schema. Moreover, strong second-order logic is usually obtainable from both by considering *saturated* (or *principal*) models, which contain all possible n-ary relations in its predicate domains. As such, adoption of the weaker semantics in our completeness proofs allows us to indirectly provide semantics also for intermediate and strong second-order logic.

The third is *constructive adequacy*. Although strong, intermediate and weak definitions are all acceptable from the perspective of classical logic, it is questionable whether this is the case from a constructive perspective. Strong second-order logics usually have a problematic relationship with the Axiom of Choice (VÄÄNÄNEN, 2021), rejected by constructive mathematicians. Intermediate second-order logic also has many properties usually deemed undesirable by constructivists, such as interdefinability of logical connectives and failure of weak normalization[44] (PRAWITZ, 2006, pg.67-73). This is not entirely unexpected, as the comprehension schema guarantees the existence of predicates satisfying some strong requirements regardless of their effective construction. Although some results due to Prawitz may be used to argue for its intuitionistic acceptability (PRAWITZ, 1970), such properties make it so that the intermediate calculus' constructive character is certainly suspicious. Since we want to provide semantics acceptable to both classical and constructive logicians (even though, as established in the first chapter, we are not adopting the constructive viewpoint), avoidance of the intermediate calculus is advisable. None of these properties are observed in the weak calculus, so acceptance of its constructive character is much less problematic.

One last problem must be faced before we proceed. The main objection raised against the recognition of weak second-order logic as properly second-order is the fact that it is so similar to first-order logic one could argue it is actually just first-order logic in disguise (MANZANO, 1996, pg. 73) (SHAPIRO, 1991). In fact, the weak logic can be embedded in first-order many-sorted logic, which in turn is embeddable in first-order logic (MANZANO, 1996). Since it only uses first-order models, the weak logic, just like many-sorted logic, could be viewed as first-order logic with a change of notation.

Are we to conclude that intermediate and strong second-order logic are the only logics properly distinct from the first-order ones, and thus the only second-order logics? The answer is no, because *the models of intermediate and strong second-order logic are also first-order models*. To see why, remember that validity in the intermediate logic is equivalent to validity in all weak models satisfying the comprehension axiom[45], and validity in the strong logic is

[44] Normalization may be recovered if one imposes additional constraints on substitutions in order to obtain ramified ("typed") second-order logic (PRAWITZ, 2006, pgs. 68-73).

[45] For a embedding of the intermediate logic into many-sorted logic, see (MANZANO, 1996, pg. 277-290). For a similar embedding into first-order logic, see (VAN DALEN, 2013, pg. 148-149).

equivalent to validity in all models with saturated predicate domains. Since weak models are first-order and validity in the intermediate and strong logics can be viewed as validity in a restricted class of weak models, validity in the intermediate and strong logics can be viewed as validity in a restricted class of first-order models. The power of such logics stems not from properties of second-order quantification, but from the fact that only first-order models of a restricted (and very strong) kind are considered. It should then not be considered a complete surprise that no standard finitary calculus can be given for the strong logic, since it is already known that validity in some simple classes of first-order models may yield properties such as undecidability[46].

The same reasons justifying the use of many-sorted logics (instead of first-order logics) and intermediate or strong second-order models (instead of restricted first-order models) also justify the use of weak second-order logic. In all cases we have mere changes in notation, but the point is that those are *interesting* changes in notation. An intuitive proof in second-order logic may not be intuitive in its first-order counterpart. The gains are in intuitive content and mathematical elegance, not expressive power.

This is also the justification van Dalen gives for the use of semantics for the intermediate logic instead of first-order models (VAN DALEN, 2013, pg. 149):

> Note that the above translation could be used as an excuse for not doing second-order logic at all, were it not for the fact that the first-order version is not nearly as natural as the second-order one. Moreover, it obscures a number of interesting and fundamental features; e.g. validity in all principal models (see below) makes sense for the second-order version, whereas it is rather an extraneous matter for the first-order version.

It is this author's opinion that weak second-order logic is the most natural extension of first-order logic, so the intermediate and strong logics should be treated either as extensions (if we are dealing with syntactic calculi) or restrictions (if we are dealing with semantic consequence) of it. Nevertheless, our proof-theoretic semantics are easily extendable to intermediate and strong second-order semantics, so those who prefer stronger logics will be rightfully contemplated.

2.4.2 Second-order natural deduction

Definition 2.4.1 Natural deduction systems for *minimal, intuitionistic and classical weak second-order logics* are obtained by using a second-order language and regarding all rules of mini-

[46] In particular, Trakhtenbrot's theorem shows that validity in the class of finite first-order models is undecidable (EBBINGHAUS; FLUM, 2006). It is still possible to give an infinitary calculus for it, however (ARRUDA; MARTINS; PEREIRA, 2012).

mal, intuitionistic and classical first-order logic as valid (respectively), plus the following rule schemas for second-order quantification:

$$\frac{\phi}{\forall X(\phi)}\,I\forall^2 \qquad \frac{\forall X(\phi)}{\phi[X/T]}\,E\forall^2 \qquad \frac{\phi[X/T]}{\exists X(\phi)}\,I\exists^2 \qquad \frac{\exists X(\phi) \qquad \begin{matrix}[\phi[X/T]] \\ \vdots \\ \psi\end{matrix}}{\psi}\,E\exists^2$$

Restriction on $I\forall^2$: X does not have free occurrences on formulas on which ϕ depends;
Restriction on $E\exists^2$: T does not have free occurrences on ψ, ϕ or any formula on which the upper occurrence of ψ depends other than $A\phi[X/T]$;

As noted before, many calculi can be given for intermediate logic, none of them as natural as the weak calculus. If one does not wish to define a new operator λ or use the comprehension schema as an axiom, substitution must be defined not only for terms, but also entire formulas (e.g. $X_0[X_0/P \vee Q]$ is a substitution yielding the formula $P \vee Q$). As mentioned before, the incompleteness results for strong second-order logic prevents us from defining a natural deduction system for it.

Definition 2.4.2 Let $\mathbb{L}$ be a second-order language. Let Γ be a set of second-order formulas and A a second-order formula of $\mathbb{L}$. The relations $\vdash^2_m$, $\vdash^2_i$ and $\vdash^2_c$ hold between them if and only if there is a deduction with premises Γ and conclusion A using only the valid rules of minimal, intuitionistic and classical second-order logic, respectively.

2.4.3 Second-order model-theoretic semantics

Models for weak minimal second-order logic are obtained by generalizing first-order definitions. We define a domain of individuals and n-ary domains for each $n \geq 0$, partially following a convention briefly suggested by Prawitz in Remarks 1 and 2 of (PRAWITZ, 1970, pgs. 263-264). Instead of containing n-ary relations, our n-ary domains contain arbitrary objects to which n-ary relations are assigned. This avoids some complications with the tracking of denotations across domains; since n-ary relations are used to evaluate predicate constants (instead of being objects named by them), a relation in $\alpha^n(k)$ might not be present in $\alpha^n(k')$ even when $k \leq k'$. Notice also that, unlike what is suggested in (PRAWITZ, 1970), we have a domain for 0-ary predicates, which is not strictly necessary but makes some aspects of models smoother. These modification also makes our semantics closer to many-sorted semantics for second-order logic (SHAPIRO, 1991, pgs. 74-76)(MANZANO; ARANDA, 2022) than to Henkin semantics (SHAPIRO, 1991, pg. 73-74), as well as closer to the proof-theoretic semantics we will later present.

Definition 2.4.3 A *generalized domain assignment function* α for a set W is a function assigning, for every $n \geq 0$, an n-ary domain $\alpha^n(k)$ to every $k \in K$ and n-ary domain $\bigcup_{k \in W} \alpha^n(k)$ to W, as well as a domain $\alpha(k)$ to every k and a domain $\bigcup_{k \in W} \alpha(k)$ to W.

Definition 2.4.4 Let $\mathbb{L}$ be a language, W a set of objects k and α a generalized domain assignment function for W. A *generalized interpretation function* β is a function assigning to every individual constant of $\mathbb{L}$ an element of $\alpha(W)$, to every n-ary predicate constant of $\mathbb{L}$ an element of $\alpha^n(W)$ and to every pair (e_n, k) of an element e_n of $\alpha^n(W)$ and an element k of W an n-ary relation on elements of $\alpha(k)$.

Definition 2.4.5 Let α be a generalized domain assignment function for W. The language $\mathbb{L}(\alpha(W))$ is obtained by adding to $\mathbb{L}$ one distinct individual constant for each element of $\alpha(W)$ and one distinct n-ary predicate constant for each of $\alpha^n(W)$.

The definition of second-order models is quite symmetric to Definition 2.3.10:

Definition 2.4.6 Let $\mathbb{L}$ be a second-order language. A *weak second-order minimal model* K for the extended language $\mathbb{L}(\alpha(W))$ is a sequence $\langle W, \leq, \alpha, \beta, v\rangle$ such that:

1. W and $\leq$ are as in Definition 2.3.1;

2. α is a generalized domain assignment function for W, satisfying the conditions that $k \leq k'$ implies $\alpha(k) \subseteq \alpha(k')$ and $\alpha^n(k) \subseteq \alpha^n(k')$ for every $n \geq 0$;

3. β is a interpretation function for $\mathbb{L}$, W and α, satisfying the conditions that $k \leq k'$ implies $\beta(\beta(P_n), k) \subseteq \beta(\beta(P_n), k')$. For every element e of $\alpha(W)$ there must be a individual constant $a \in \mathbb{L}(\alpha(W))$ such that $\beta(a) = e$, for every element e_n of $\alpha^n(W)$ there must be a n-ary predicate constant $P_n \in \mathbb{L}(\alpha(W))$ such that $\beta(P_n) = e_n$, and for every k there must be some $e_0 \in \alpha^0(k)$ such that $\beta(e_0) = \bot$;

4. v is as in Definition 2.3.1 for 0-ary predicate constants, but satisfying the condition that $\beta(P_0) \notin \alpha^0(k)$ implies $v(P_0, k) = \emptyset$. For predicates of arity greater than 0, $\beta(P_n) \notin \alpha^n(k)$ implies $v(P_n(a^1, ..., a^n), k) = \emptyset$, and $\beta(P_n) \in \alpha^n(k)$ implies $v(P_n(a^1, ..., a^n), k) = T$ iff $\langle \beta(a^1), \ldots, \beta(a^n)\rangle \in \beta(\beta(P_n), k)$[47].

Definition 2.4.7 Let $\mathbb{L}$ be a second-order language and K a weak second-order minimal model. The relations $\vDash_k^K$ and $\vDash^K$ are defined for the sentences of $\mathbb{L}(\alpha(W))$ as follows:

1. Clauses 1 through 9 are the same as in Definition 2.3.2. Clauses 10 and 11 are the same as in definition 2.3.11;

[47] Just like in the first-order case, from the definition of β it follows that every atomic formula satisfies the heredity property.

12. $\models_k^K \forall X_n(A) \iff \forall k'(k \leq k') : \models_{k'}^K A[X_n/P_n]$, for all P_n such that $\beta(P_n) \in \alpha^n(k)$;

13. $\models_k^K \exists X_n(A) \iff \forall k'(k \leq k') : \models_{k'}^K A[X_n/P_n]$, for some P_n such that $\beta(P_n) \in \alpha^n(k)$;

The clauses are defined only for sentences, but we may once again extend them to formulas via Definition 2.3.12. Just like in propositional and first-order semantics, intuitionistic and classical second-order models are defined by imposing the conditions of Definitions 2.3.3 and 2.3.4 on minimal second-order models.

Aside from being a multi-sorted semantics, this presentation differs from the usual ones because second-order languages may contain any number of predicate constants. Since first-order languages are special cases of second-order languages, the traditional "full languages" may be obtained by switching back to first-order languages.

Second-order weak semantic consequence may be defined as follows:

Definition 2.4.8 Let $\mathbb{L}$ be a second-order language. Let Γ be a set of second-order formulas and A a second-order formula of $\mathbb{L}$. The relations $\models_m^2$, $\models_i^2$ and $\models_c^2$ hold between them if and only if $\Gamma \models^K A$ holds for every minimal, intuitionistic and classical second-order model K, respectively.

As for intermediate and strong second-order logic, we require satisfaction of the following properties:

Definition 2.4.9 A weak minimal second-order model K is also an *intermediate minimal second-order model* if, for all A, $\models_k^K \exists X \forall x^1 \ldots \forall x^n (X x^1 \ldots x^n \leftrightarrow A)$ holds, provided X does not have free occurrences in A.

Definition 2.4.10 A weak minimal second-order model K is also a *strong minimal second-order model* if, for all $n \geq 1$ and all k, if R is a n-ary relation on $\alpha(k)$ then there is a $e_n \in \alpha^n(k)$ such that $\beta(e_n, k) = R$.

Intermediate models must satisfy all instances of the comprehension schema, and each n-ary domain of strong models must contain all n-ary relations on its 0-ary domain. We may once again obtain intuitionistic and classical versions of such models by imposing the constraints of Definition 2.3.3 and 2.3.4. Semantic consequence relations for these logics may also be obtained by a straightforward adaptation of Definition 2.4.8.

2.5 Atomic bases

Before proceeding to the next chapter, we define the notion of *atomic bases* (also called *atomic systems* or just *bases*), thoroughly used in Chapter 3. Bases are sets of atomic rules, defined as natural deduction rules (cf. Definition 2.2.1) in which the premises and conclusion are atomic sentences (cf. Definitions 2.1.8 and 2.1.12).

Atomic bases figure prominently in proof-theoretic validity and base-extension semantics, two proof-theoretic semantics in which validity in bases is used to define validity in a broader logical system. Intuitively, bases contain rules one is inclined to accept as valid, but not logically valid. They may thus be regarded as *inferential knowledge bases*, specifying which non-logical rules one can justifiably make. Logical validity may then be defined by considering validity in bases and their possible *extensions* – that is, new bases obtainable from them by regarding new atomic rules as valid.

To exemplify, consider the following rule:

$$\frac{\text{It is sunny}}{\text{It is light}} \qquad \text{or} \qquad \frac{S_0}{L_0}$$

Many would regard it as valid, but not as logically valid. Consider now a base S which regards this rule as the only valid one. Clearly, every extension S' of S allowing us to categorically affirm that it is sunny – that is, allowing us to give a closed deduction of S_0 – also allows us to categorically affirm "It is light" by giving a closed deduction of L_0, since every rule of S is contained in its extension S' and we may apply this rule at the end of the closed deduction of S_0 to obtain a new closed deduction of L_0 in S'. If we define "It is A, therefore it is B" as holding in a base whenever, for every extension of it, if there is a closed deduction of A there is also one of B, then we may conclude that "It is S_0, therefore it is L_0" ("It is sunny, therefore it is light") holds in S.

2.5.1 Standard bases

We start by defining bases themselves:

Definition 2.5.1 An *atomic rule* is a natural deduction rule in which the premises, the conclusion and the formulas in discharged sets are atomic sentences.

Definition 2.5.2 A *atomic base* is a set of atomic rules.

Bases may also be classified according to the complexity of their rules:

Definition 2.5.3 A *atomic axiom* (or *axiomatic rule*) is a atomic rule with an empty sequence of premise.

Definition 2.5.4 A *production rule* is a atomic rule discharging no formulas.

Definition 2.5.5 A *axiomatic base* is a set of atomic axioms.

Definition 2.5.6 A *production base* is a set of production rules.

Axiomatic bases are particular cases of production bases, which in turn are particular cases of atomic bases. This classification becomes relevant whenever we want to exclude the possibility of hypothetical judgments, and thus of inferential discharge.

We also define a few auxiliary notions:

Definition 2.5.7 A base S' is an *extension*[48] of a base S if $S \subseteq S'$.

Definition 2.5.8 Let Γ be a (possibly empty) set of atomic sentences and A a atomic sentence. $\Gamma \vdash_S A$ holds if there is a deduction of A from Γ using only the rules of S. We call any such deduction a *atomic deduction*.

Definition 2.5.9 A base S is *consistent* if $\nvdash_S \bot$.

Definition 2.5.10 Propositional and first-order bases are those in which all sentences occurring on the base's rules are propositional and first-order sentences, respectively[49].

We also define the notion of *composition of deductions*. It is commonly defined for all operators and used to prove normalization results (PRAWITZ, 2006), but we define only for atomic deductions.

Definition 2.5.11 A *composition* of a deduction Π with conclusion A and a deduction Π' depending on Γ with conclusion B is a deduction Π'' obtained by putting one copy of the deduction Π above every undischarged A occurring in Π' (if any).

To exemplify, consider the following deductions with rules labelled 1, 2, 3 and 4:

$$\Pi = \cfrac{\cfrac{P}{Qab}\,1 \qquad Qba}{Rabc}\,2 \qquad\qquad \Pi' = \cfrac{Rabc \qquad \cfrac{}{S}\,3}{Qed}\,4 \qquad\qquad \Pi'' = \cfrac{\cfrac{\cfrac{P}{Qab}\,1 \qquad Qba}{Rabc}\,2 \qquad \cfrac{}{S}\,3}{Qed}\,4$$

Π'' is a composition of Π with Π'. From the definition it follows that, if Π depends on Δ and Π' depends on Γ, Π'' depends on $\Delta \cup \{\Gamma - \{A\}\}$, and also that, if Π' does not depend on A, then by composing a deduction Π concluding A with it the Π'' is Π' itself.

[48] This definition uses a minor variation of the notation in (PIECHA; SCHROEDER-HEISTER, 2016).
[49] There is no need to go beyond first-order, since all second-order sentences are also first-order sentences.

2.5.2 Higher-order bases

We may also use higher-order natural deduction systems (SCHROEDER-HEISTER, 1984) to define atomic systems containing atomic rules capable of discharging other atomic rules. This requires not only a generalization of atomic bases, but also of the natural deduction definitions given in Section 2.2. Higher-order rules are typically defined by assigning *levels* to rules and modifying Definition 2.2.1 so as to allow rules of level l to discharge sets containing rules of lesser level. The notion of rule schema is likewise modified, so rules schemas of level l are instantiated by rules of level l.

Using the same notation as in Definition 2.2.1, we define[50] (higher-order) natural deduction as follows:

Definition 2.5.12 *Higher-order rules* are defined as follows:

1. A natural deduction rule with a empty sequence of premises (cf. Definition 2.2.1 and the comments that follow it) is a higher-order rule of level 0;

2. For levels greater than 0, let $\langle A^1, \dots, A^n \rangle$ be a sequence of n formulas, $\langle \Omega^1, \dots, \Omega^n \rangle$ a sequence of n sets of higher-order rules with maximum level l and B a formula. Then:

$$\dfrac{\begin{matrix}[\Omega^1] \\ \vdots \\ A^1\end{matrix} \quad \dots \quad \begin{matrix}[\Omega^n] \\ \vdots \\ A^n\end{matrix}}{B}$$

Is a *higher-order rule* of level $l+1$. We represent this in horizontal notation by $[\Omega^1 \Rightarrow A^1, \dots, \Omega^n \Rightarrow A^n /^H B]$, and $\Omega^m \Rightarrow A^m$ is written as A^m when Ω^m is empty.

The definition of deduction may be adapted as follows:

[50] Our definitions slightly diverge from those given by Schroeder-Heister, as do our definitions of higher-order atomic bases from authoritative sources such as (PIECHA, 2016). The main difference is that, in our definition, the premises of higher-order rules are formulas, whereas in traditional definitions they are rules of lesser level. We still get as a result that only rule assumptions may be used in deductions (since it follows from our definitions that a formula can only occur in a deduction as a consequence of a rule), but this change allows our notation to be somewhat closer to that of standard natural deduction. A consequence of this is that the most natural representation of our definitions are in tree form, while the original ones are arguably more natural when spelled out in horizontal notation.
It should also be noted that our definitions lead to a slight difference on how the level of rules is treated: in the original definition, every rule of level l can only discharge rules whose level is at most $l-2$, while our rules are allowed to discharge rules whose level is at most $l-1$. This is so because the traditional rules of level 1 (which do not discharge rules) and 2 (which discharge only rules of level 0) both become particular cases of our rules of level 1.

Definition 2.5.13 A *higher-order deduction* (or *higher-order derivation*) of a formula A depending on a set Γ of higher-order rules, denoted by $\begin{array}{c}\Gamma\\ \Pi\\ A\end{array}$, is defined as follows:

1. If R^0 is a rule of level 0 with conclusion A, R^0 is a deduction of A depending on $\Omega \cup \{R^0\}$, for any set Ω of higher-order rules;

2. If $\begin{array}{c}\Gamma^m\\ \Pi^m\\ A^m\end{array}$ is a higher-order deduction of A^m depending on Γ^m for $(1 \leq m \leq n)$ and

$$\frac{\begin{array}{c}[\Omega^1]\\ \vdots\\ A^1\end{array} \quad \ldots \quad \begin{array}{c}[\Omega^n]\\ \vdots\\ A^n\end{array}}{B}$$

is a higher-order natural deduction rule R, then

$$\frac{\begin{array}{c}\Gamma^1, [\Omega^1]\\ \Pi^1\\ A^1\end{array} \quad \ldots \quad \begin{array}{c}\Gamma^n, [\Omega^n]\\ \Pi^n\\ A^n\end{array}}{B}$$

is a higher-order deduction of B depending on the set $R \cup (\bigcup_{1 \leq k \leq n} \Gamma^k - \Omega^k)$ of higher-order rules.

Validity of deductions can no longer be defined as in Definition 2.2.9, since in order to entertain the idea of rule assumptions we must allow deductions of a logic to manipulate rules not considered valid by that logic. Instead of defining validity for each logic, we go straight to valid syntactic consequence relations:

Definition 2.5.14 Let Δ be the set of all rules regarded as valid by a logic L. Then $\Gamma \vdash_L A$ holds if and only if there is a higher-order deduction of A depending on $\Delta \cup \Gamma$.

The intuition behind this is that, when higher-order natural deduction is considered, the definition of deduction in which every possible rule can be used is shared by all logics, but the rules considered valid by a particular logic can be "taken for granted" and excluded from the set of rules that are genuine assumptions.

Higher-order versions of definitions for standard bases may be given as follows:

Definition 2.5.15 A *higher-order atomic rule* is a higher-order rule with atomic sentences for premises and conclusion and only higher-order atomic rules in its sets of discharged rules.

Definition 2.5.16 A *higher-order atomic deduction* is a deduction that uses only higher-order atomic rules (even for discharged rules).

Definition 2.5.17 A *higher-order base* is a set of higher-order rules.

Definition 2.5.18 Let Γ be a (possibly empty) set of higher-order atomic rules and A an atomic sentence. $\Gamma \vdash_S A$ holds iff there is a higher-order atomic deduction of A depending on $S \cup \Gamma$.

The reason we define higher-order systems in a separate section instead of using them from the start is that, even though many results on the proof-theoretic literature hold in greater generality when higher-order systems are considered, this does not seem to be the case for the semantics we present. Differences may arise when particular cases are considered, but the general cases do not seem to be affected by their use. We therefore opt to use standard natural deduction for the sake of simplicity, mentioning higher-order systems only in order to briefly comment on some of their relations with other systems and some particular cases in which differences might appear.

3 MULTIBASE SEMANTICS

As seen in Section 1.2.3, the concept of proof has both *categorical* and *hypothetical* aspects. Categorical proofs are those in which statements must be proved directly, as opposed to proved conditional on some other statement being proved. This chapter introduces a proof-theoretic semantics in which proofs are initially defined categorically, but then proven to also have important hypothetical characteristics.

The semantics here considered is very similar to *generalized proof-theoretic validity* (STAFFORD; NASCIMENTO, 2023), and also to *base-extension semantics* (SANDQVIST, 2015). In fact, our main completeness result for propositional logic is essentially the same as the result for generalized proof-theoretic validity[51], but we also provide new propositional results and new generalizations to first-order and second-order logics.

3.1 **Proof-theoretic validity**

3.1.1 Original definitions

Proof-theoretic validity is a proof-theoretic semantics due to Prawitz (with simplifications by Schroeder-Heister) in which the semantic content of formulas is determined by closed canonical proofs in natural deduction (PRAWITZ, 1973)(SCHROEDER-HEISTER, 2006). This can be defined through recourse to atomic bases: an atom is valid in a base if it is possible to give a closed proof of it using only its rules, validity for non-atomic formulas being given either by the introduction rules of logical connectives (rules $I\wedge$, $I\vee^1$, $I\vee^2$ and $I\rightarrow$ of Definition 2.2.10)[52] or semantic clauses inspired by them. In other words, atomic rules allow us to obtain categorical proofs of atoms in a given base, which may be used together with either introduction rules or semantic clauses to obtain categorical proofs of the connectives in that same base. Logical validity is then defined as validity in all possible atomic bases.

Since atomic sentences will be used extensively in this section, we temporarily fix the

[51] I am eternally grateful to Will Stafford for letting me coauthor the paper in which those results were first presented. They were discovered independently, but he was the first one to find them. Though he had every right to publish as the sole author, he gracefully suggested we coauthor a paper to present the results together. Due to the independent discovery, however, there were two very different notations for essentially the same semantic framework; his notation was used in (STAFFORD; NASCIMENTO, 2023), mine is used in this thesis. The new notation is introduced mainly because some of the new results seem more natural in it.

[52] The *canonical proofs* mentioned earlier are proofs ending with an application of an introduction rule.

notation $\{a, b, c, ...\}$ for propositional atoms[53], to be used whenever there is no risk of confusion with individual constants.

Semantic clauses for proof-theoretic validity are given by Piecha, Sanz and Schroeder-Heister in (PIECHA; SCHROEDER-HEISTER, 2016) and (PIECHA; SANZ; SCHROEDER-HEISTER, 2015):

Definition 3.1.1 S-validity ($\vDash_S$) and proof-theoretic validity ($\vDash$) are defined as follows:

1. $\vDash_S a \iff \vdash_S a$, for atomic a;
2. $\vDash_S A \wedge B \iff \vDash_S A$ and $\vDash_S B$;
3. $\vDash_S A \vee B \iff \vDash_S A$ or $\vDash_S B$;
4. $\vDash_S A \to B \iff A \vDash_S B$;
5. $\vDash_S \Gamma \iff \{\vDash_S A_i \mid A_i \in \Gamma\}$, where Γ is a set of formulas;
6. $\Gamma \vDash_S A \iff \forall S'(S \subseteq S') : \vDash_{S'} \Gamma$ implies $\vDash_{S'} A$;
7. $\Gamma \vDash A \iff \forall S : \Gamma \vDash_S A$.

These clauses are similar to those in Definition 2.3.2, but they differ both because atoms are now evaluated according to their derivability in bases and because clause 6 refers to all extensions S' of S instead of using accessibility relations of models.

Prawitz had conjectured (PRAWITZ, 1971) that proof-theoretic validity (plus a particular treatment of $\bot$) would be complete with respect to derivability in intuitionistic logic, in the sense that $\Gamma \vDash A$ plus a clause excluding the possibility of a proof of $\bot$ would imply $\Gamma \vdash_i^0 A$. We could also reasonably expect that the above clauses (without any special treatment for $\bot$) would be complete with respect to $\vdash_0^m$, since they do seem to reflect both the intuitive meaning of logical constant and the rules of natural deduction. Sadly, this was proven not to be the case in (PIECHA; SANZ; SCHROEDER-HEISTER, 2015), and it was later shown that proof-theoretic validity is complete with respect to a stronger logic called *general inquisitive logic*[54] (STAFFORD, 2021).

On the other hand, completeness results were obtained by Goldfarb and Sandqvist for deviant definitions of validity (GOLDFARB, 2016)(SANDQVIST, 2009). Though very interesting, Sandqvist's results use a hypothetical definition of proof, and Goldfarb's use a semantic framework significantly different from the standard ones.

[53] The clause in Definition 2.3.2 does not use this notation because then they could not be extended to first and second-order logic without adaptations.

[54] Since it is not closed under substitution, general inquisitive logic would not be considered a logic by standard definitions of logicality. We argue this should be taken as yet another argument in favour of logicality-free definitions of logic.

3.1.2 Generalized proof-theoretic validity

Completeness results for generalized proof-theoretic validity – a framework very similar to proof-theoretic validity – were given in (STAFFORD; NASCIMENTO, 2023). The main difference between proof-theoretic validity and generalized proof-theoretic validity is that, in the latter, we fix a *proof-theoretic system* and evaluate formulas in a base by considering only extensions contained inside that particular proof-theoretic system, as opposed to considering all its possible extensions.

Definition 3.1.2 A base is *explosive* if it contains a rule $[\bot/a]$ for every atom a (cf. the horizontal notation of Definition 2.2.1).

Definition 3.1.3 A *minimal proof-theoretic system* $\mathfrak{G}$ is a set of bases.

Definition 3.1.4 A *intuitionistic proof-theoretic system* $\mathfrak{G}$ is a set of explosive bases.

Definition 3.1.5 The relations of weak system validity ($\vDash^{\mathfrak{G}}_S$), system validity ($\vDash^{\mathfrak{G}}$) and generalized proof-theoretic validity ($\vDash'$) are defined as follows, for $S \in \mathfrak{G}$:

1. $\vDash^{\mathfrak{G}}_S a \iff \vdash_S a$, for atomic a;
2. $\vDash^{\mathfrak{G}}_S A \land B \iff \vDash^{\mathfrak{G}}_S A$ and $\vDash^{\mathfrak{G}}_S B$;
3. $\vDash^{\mathfrak{G}}_S A \lor B \iff \vDash^{\mathfrak{G}}_S A$ or $\vDash^{\mathfrak{G}}_S B$;
4. $\vDash^{\mathfrak{G}}_S A \to B \iff A \vDash^{\mathfrak{G}}_S B$;
5. $\vDash^{\mathfrak{G}}_S \Gamma \iff \{\vDash^{\mathfrak{G}}_S A_i \mid A_i \in \Gamma\}$, where Γ is a set of formulas;
6. $\Gamma \vDash^{\mathfrak{G}}_S A \iff \forall S'(S' \in \mathfrak{G}$ and $S \subseteq S') : \vDash^{\mathfrak{G}}_{S'} \Gamma$ implies $\vDash^{\mathfrak{G}}_{S'} A$;
7. $\Gamma \vDash^{\mathfrak{G}} A \iff \forall S \in \mathfrak{G} : \Gamma \vDash^{\mathfrak{G}}_S A$.
8. $\Gamma \vDash' A \iff \forall \mathfrak{G} : \Gamma \vDash^{\mathfrak{G}} A$.

The completeness results in (STAFFORD; NASCIMENTO, 2023) are proved for intuitionistic propositional logic and intuitionistic proof-theoretic systems, but (as will be shown) it is straightforward to adapt the proof so that it shows completeness for minimal propositional logic and minimal proof-theoretic systems.

Since proof-theoretic systems fix a set of possible extensions just like Kripke models fix a set of worlds, generalized proof-theoretic validity is even closer to Kripke semantics than standard proof-theoretic validity. The main difference is that we may use any partial order on W in models, whereas generalized proof-theoretic validity always uses the extension relation. Generalized proof-theoretic validity also differs from standard proof-theoretic validity by quantifying over proof-theoretic systems to define general validity (as opposed to quantifying over bases), so the notion of S-validity is lost.

3.2 Multibases

We now introduce the semantic framework that will be investigated in this thesis, called *multibase semantics*.

3.2.1 Basic definitions

We start by defining the following notions:

Definition 3.2.1 A *multibase* is any non-empty sequence of bases.

Definition 3.2.2 An atomic base S' is an *extension of the base S in the multibase M* (written $S \subseteq_M S'$) if and only if $S \subseteq S'$ and both S and S' are in the multibase M.

Just as in the case of generalized proof-theoretic validity, validity in a base can be defined more narrowly by considering only their extensions contained in a multibase. Multibases can be thought of as collection of bases and their "admissible evolutions", in the sense that every element of the sequence is a possible state of inferential knowledge and every extension of a base in the same multibase is a possible extension of that state of inferential knowledge. Bases occurring in a multibase do not necessarily have any kind of relation with their successors or predecessors, which can be interpreted as the possibility of always introducing a completely new state of knowledge.

For now we consider only propositional bases (cf. Definition 2.5.10). First-order bases will be left for the section dealing with predicate logic.

Definition 3.2.3 The relations of base validity ($\Vdash_S^M$), multibase validity ($\Vdash^M$) and standard validity ($\Vdash$) are defined as follows, for $S \in M$:

1. $\Vdash_S^M a \iff \vdash_S a$, for atomic a;
2. $\Vdash_S^M A \wedge B \iff \Vdash_S^M A$ and $\Vdash_S^M B$;
3. $\Vdash_S^M A \vee B \iff \Vdash_S^M A$ or $\Vdash_S^M B$;
4. $\Vdash_S^M A \to B \iff A \Vdash_S^M B$;
5. $\Vdash_S^M \Gamma \iff \{\Vdash_S^M A_i \mid A_i \in \Gamma\}$, where Γ is a set of formulas;
6. $\Gamma \Vdash_S^M A \iff \forall S'(S \subseteq_M S') : \Vdash_{S'}^M \Gamma$ implies $\Vdash_{S'}^M A$;
7. $\Gamma \Vdash^M A \iff \forall S \in M : \Gamma \Vdash_S^M A$.
8. $\Gamma \Vdash A \iff \forall M : \Gamma \Vdash^M A$.

This is clearly a mere restatement of the clauses for generalized proof-theoretic validity, the only difference being that now we use sequences instead of sets. It is not clear what the benefits of using sequences are when we consider general validity for multibases, but this is no longer the case when we transition to *focused validity*.

3.2.2 Focused multibases

Focused multibases are defined as follows:

Definition 3.2.4 A *focused multibase* F for S^1 is any multibase $\langle S^1, S^2, ...\rangle$ such that, for all $n > 1$ (if any), there is an S^m with $m < n$ such that $S^m \subseteq S^n$.

Intuitively, we now fix an initial state of knowledge and consider only its possible developments inside a fixed frame of possibilities, not allowing new states of knowledge not related to S^1 to occur in the sequence. Every S^n expands some previous state of knowledge S^m and, due to transitivity of the extension relation, also S^1.

Focused multibases may also be defined by considering only the initial base:

Corollary 3.2.5 F is a focused multibase for $S^1 \iff F$ is of shape $\langle S^1, S^2, ...\rangle$ and, for all $n > 1$ (if any), $S^1 \subseteq_F S^n$.

Proof:

($\Rightarrow$): Let S^k be any base on the focused multibase F. By definition, there must be a $j < k$ such that $S^j \subseteq_F S^k$. This also holds for S^j, so there must be a S^i with $i < j$ such that $S^i \subseteq_F S^j$. Since the number is always decreasing, we eventually reach a S^m such that $S^1 \subseteq_F S^m$ and, by transitivity of extension, we conclude $S^1 \subseteq_F S^k$.

($\Leftarrow$): If $S^1 \subseteq_F S^k$ for all $S^k \in F$ the result is immediate, since $1 < k$. □

Focused multibases for S may also be called multibases focused on S, or simply focused multibases when the particular S is not relevant.

Focused multibases enjoy many properties that standard multibases lack. For instance, we can recover the notion of S-validity as follows:

Definition 3.2.6 A focused multibase F for S is *saturated* if $S \subseteq_F S'$ implies $S' \in F$.

Proposition 3.2.7 Let F^* be a saturated focused multibase for S. Then $\Gamma \vDash_S A$ holds (cf. Definition 3.1.1) if and only if $\Gamma \Vdash_S^{F^*} A$ holds.

Corollary 3.2.8 $\Gamma \vDash A$ holds if and only if, for all S, we have that $\Gamma \Vdash_S^{F^*} A$ holds for some saturated focused multibase F^* for S.

Proof: Notice that $\vDash_S$ is defined by taking into account all extensions of S, and so is $\Vdash_S^{F^*}$. This leads to a collapse between clause 6 of both definitions, which in turn leads to a collapse between clauses 1 through 5. □

Aside from recovering the original notion of S-validity and proof-theoretic validity, we can provide new notions of *generalized S-validity* and *focused validity*:

Definition 3.2.9 Generalized S-validity ($\Vdash_S$) and focused validity ($\Vdash_m^0$) are defined as follows:

1. $\Gamma \Vdash_S A$ iff $\Gamma \Vdash^F A$ for all focused multibases F for S.
2. $\Gamma \Vdash_m^0 A$ holds iff $\Gamma \Vdash_S A$ for all S.

Notice that $\Gamma \Vdash_m^0 A$ holds iff $\Gamma \Vdash_S A$ for all S and $\Gamma \Vdash_S A$ holds iff $\Gamma \Vdash^F A$ for all F focused on S, so it follows that $\Gamma \Vdash_m^0 A$ holds iff $\Gamma \Vdash^F A$ holds for all multibases F focused on some S (that is, all focused multibases).

Since generalized S-validity is defined as validity in all focused multibases for S and S-validity is equivalent to validity in saturated focused multibases for S, S-validity is a special case of generalized S-validity. Focused validity is then defined as validity in all multibases focused on S for every S, which is much closer to proof-theoretic validity than standard multibase validity and generalized proof-theoretic validity – since they quantify over multibases and proof-theoretic systems instead of bases.

This change is not merely aesthetic: focused validity has properties that standard multibase validity, generalized proof-theoretic validity and even semantic consequence for Kripke models lack. Many properties that differentiate proof-theoretic semantics from model-theoretic semantics are a direct consequence of quantification over bases and variants of S-validity. This is not to say that *all* interesting properties of proof-theoretic semantics are a consequence of variants of S-validity, however; as will be shown, some features of multibase semantics for predicate logic that are entirely absent in Kripke models are present even in standard multibases.

Notice that, when defining focused multibases, we do not forbid extensions of a base to precede it in the sequence, so validity in a base might depend on validity in its predecessors. From the fact that a new state of knowledge is obtained by extending any previous one it does not follow that there is no previously obtained state of knowledge stronger than it. There might also be repetitions of bases. This is corrected if the sequence is built by ordering the bases appropriately and deleting repetitions, but we do not impose this as a formal requirement to simplify some proofs.

Although we are dealing only with multibases, similar notions may be obtained in generalized proof-theoretic validity as follows:

Definition 3.2.10 A multibase system $\mathfrak{G}/S$ is a proof-theoretic system $\mathfrak{G}$ such that $S \subseteq S'$ holds for all $S' \in \mathfrak{G}$.

Focused proof-theoretic systems for S may be defined as proof-theoretic systems with S as its fixed point, and all other focused notions may be adapted accordingly.

It is not entirely clear how one would justify the use of proof-theoretic systems with fixed points instead of general proof-theoretic systems, whereas the interpretation we gave of focused multibases makes it quite natural to give special treatment to the first element of the sequence. This is, in fact, a purely aesthetic matter, but sequences are a better fit for some intuitions than set-theoretical formulations. Some proofs also seem to become more intuitive when sequences are used. Other philosophical aspects of focused multibases that may be considered of interest will also be presented in section 3.6.2.

3.2.3 Soundness and completeness for propositional multibases

Standard validity and focused validity for multibases with propositional bases are sound and complete with respect to $\vdash^0_m$ (cf. Definition 2.2.13), and this holds even if we consider only finite multibases. We start by showing that the notions of standard validity and minimal propositional semantic consequence (cf. Definitions 2.3.2 and 2.3.5) are equivalent, then showing how this results in soundness and completeness for our notions.

We refer to multibases containing only propositional bases simply as "multibases" for now; unrestricted multibases will be used only when dealing with predicate logic.

Definition 3.2.11 For any multibase M, its corresponding Kripke minimal model K^M (cf. Definition 2.3.1) is defined as follows:

1. The set W of K^M is the set of all bases occurring in M;
2. $S \leq S'$ if and only if $S \subseteq_M S'$.
3. For any atom a, $v(a, S) = T$ if and only $\vdash_S a$

Since every rule in a base S is contained in all its extensions S', every closed argument showing $\vdash_S a$ is also a closed argument showing $\vdash_{S'} a$, which guarantees that the heredity condition is satisfied. It also follows from the fact that $\subseteq_M$ induces a partial order that $\leq$ is also a partial order (and thus that this is indeed a Kripke model).

Lemma 3.2.12 $\vDash^{K^M}_S A \Longleftrightarrow \Vdash^M_S A$.

Proof: The result is immediate for atomic formulas. For other formulas, we prove it via induction on formulas:

1. $\vDash^{K^M}_S A \wedge B$. Then $\vDash^{K^M}_S A$ and $\vDash^{K^M}_S B$. Induction hypothesis: $\Vdash^M_S A$ and $\Vdash^M_S B$. Therefore, $\Vdash^M_S A \wedge B$. For the other direction, let $\Vdash^M_S A \wedge B$. Therefore, $\Vdash^M_S A$ and $\Vdash^M_S B$. Induction hypothesis: $\vDash^{K^M}_S A$ and $\vDash^{K^M}_S B$. Therefore, $\vDash^{K^M}_S A \wedge B$.

2. $\vDash_S^{KM} A \vee B$. Then either $\vDash_S^{KM} A$ or $\vDash_S^{KM} B$. Induction hypothesis: either $\Vdash_S^M A$ or $\Vdash_S^M B$. In any case we have $\Vdash_S^M A \vee B$. For the other direction, let $\Vdash_S^M A \vee B$. Then either $\Vdash_S^M A$ or $\Vdash_S^M B$. Induction hypotheses: either $\vDash_S^{KM} A$ or $\vDash_S^{KM} B$. In any case we have $\vDash_S^{KM} A \vee B$.

3. $\vDash_S^{KM} A \rightarrow B$. Then $A \vDash_S^{KM} B$, and so $\vDash_{S'}^{KM} A$ implies $\vDash_{S'}^{KM} B$ for any $S \leq S'$. Induction hypothesis: since $S \leq S'$ iff $S \subseteq_M S'$ by construction, $\Vdash_{S'}^M A$ implies $\Vdash_{S'}^M B$ for any $S \subseteq_M S'$. Therefore $A \Vdash_S^M B$, and so $\Vdash_S^M A \rightarrow B$. For the other direction, let $\Vdash_S^M A \rightarrow B$. Then $A \Vdash_S^M B$, and so $\Vdash_{S'}^M A$ implies $\Vdash_{S'}^M B$ for any $S \subseteq_M S'$. Induction hypothesis: since $S \subseteq_M S'$ iff $S \leq S'$ by construction, $\vDash_{S'}^{KM} A$ implies $\vDash_{S'}^{KM} B$ for any $S \leq S'$. Therefore $A \vDash_S^{KM} B$, and so $\vDash_S^{KM} A \rightarrow B$. $\square$

Corollary 3.2.13 $\Gamma \vDash^{KM} A \iff \Gamma \Vdash^M A$.

Theorem 3.2.14 $\Gamma \vDash_m^0 A$ implies $\Gamma \Vdash A$

Proof: Assume $\Gamma \vDash_m^0 A$. Then $\Gamma \vDash^K A$ for any minimal propositional model K. Let M be any multibase. By Corollary 3.2.13 we have $\Gamma \vDash^{KM} A \iff \Gamma \Vdash^M A$. Since $\Gamma \vDash^K A$ holds for all minimal propositional K, we have $\Gamma \vDash^{KM} A$, thus $\Gamma \Vdash^M A$. $\square$

The other direction is trickier. The extension relation is entirely defined by the contents of bases, so we are not free to decide which base extends which. We can, however, construct bases in such a way that extensions work exactly as intended.

Definition 3.2.15 A *vacuous rule* is an atomic rule concluding an atomic sentence from itself (e.g. $[a/a]$, $[b/b]$)

Vacuous rules add absolutely nothing to the deductive power of a base, but they are rules nevertheless. They can be used to induce precisely the structure we want.

We briefly show that they are deductively void as follows:

Lemma 3.2.16 If there is a deduction in S, then there is a deduction in S using no vacuous rules.

Proof: The following procedure eliminates vacuous rules:

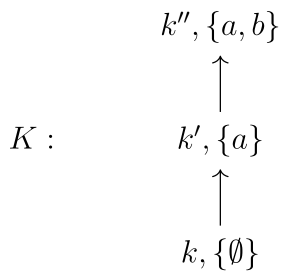
$$\begin{array}{c}\Pi\\ \dfrac{a}{a}\\ \Pi'\\ b\end{array} \quad \text{is transformed into} \quad \begin{array}{c}\Pi\\ a\\ \Pi'\\ b\end{array}$$

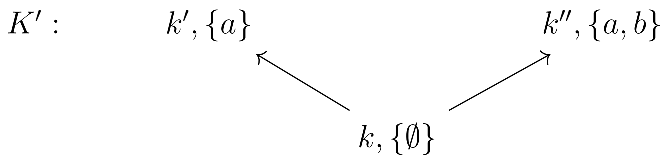
We repeat the procedure until all vacuous rules are removed to show the result. □

Now consider, for example, two models K and K' with the following assignments:

$v(k,a) = \emptyset\ ,\ v(k,b) = \emptyset\ ,\ v(k',a) = T\ ,\ v(k',b) = \emptyset\ , v(k'',a) = T\ ,\ v(k'',b) = T$

Let $k \leq k'$ and $k' \leq k''$ be the only relations holding in K, and let $k \leq k'$ and $k \leq k''$ be the only ones holding in K' (relations induced by partial ordering also hold implicitly). If we briefly fix the notation $k, \{a, ..., a^n\}$ to represent the fact that only the atoms $a, ..., a^n$ are assigned T in k, the resulting models are as follows:





In both cases, the multibase obtained by adding a atomic axiom $[/a]$ to some S for every $v(a,k) = T$ is as follows:

$M = \langle S = \{\emptyset\},\ S' = \{[/a]\},\ S'' = \{[/a],[/b]\}\rangle$

Since the final set $\{[/a],[/b]\}$ is an extension of the previous set $\{[/a]\}$, $S' \subseteq_M S''$ necessarily holds, thus the multibase is isomorphic only to K.

This may be remedied by the use of vacuous rules, which allow us to generate the following multibases by assigning a vacuous rule to each k:

$M = \langle S = \{[a/a]\},\ S' = \{[/a],[a/a],[b/b]\},\ S'' = \{[/a],[/b],[a/a],[b/b],[c/c]\}\rangle$
$M' = \langle S = \{[a/a]\},\ S' = \{[/a],[a/a],[b/b]\},\ S'' = \{[/a],[/b],[a/a],[c/c]\}\rangle$

We have both $S \subseteq_M S'$ and $S' \subseteq_M S''$, since the vacuous rule $[b/b]$ of S' is contained in S''. However, $S' \subseteq_{M'} S''$ does not hold, as S'' does not contain $[b/b]$ and thus is not an extension of S'. Vacuous rules thus give us full control over extensions.

Definition 3.2.17 For any minimal model K, a corresponding multibase M^K for it is defined as follows:

1. The sequence of M contains one system S^k for each $k \in W$.
2. If $v(a, k) = T$, we add to S^k the atomic axiom concluding a.
3. If $k \leq k'$, we add the vacuous rule $[a^k/a^k]$ to $S^{k'}$.
4. Each S^k contains no rules other than those added by the above procedures.

Lemma 3.2.18 $k \leq k'$ if and only if $S^k \subseteq_{M^K} S^{k'}$.

Proof: Assume $k \leq k'$. Then S^k and $S^{k'}$ are constructed by the addition of axioms rule and vacuous rules for k and k', respectively. Due to the heredity condition, if $v(a, k) = 1$ then $v(a, k') = 1$, hence all atomic axioms in S^k are also in $S^{k'}$. Now assume that there is some vacuous rule $[a^{k''}/a^{k''}]$ in k. Then $k'' \leq k$ and, due to transitivity of $\leq$ and our assumption, $k'' \leq k'$ holds, hence the vacuous rule is also in k'. Since all atomic axioms and vacuous rules of S^k are in $S^{k'}$, we conclude $S^k \subseteq_{M^K} S^{k'}$.

For the other direction, let $S^k \subseteq_{M^k} S^{k'}$. Since every vacuous rule $[a^{k''}/a^{k''}]$ in S^k is also in $S^{k'}$, by the structure of the procedure for adding vacuous rules we conclude that $k'' \leq k$ implies $k'' \leq k'$, for every k''. By reflexivity of $\leq$ we have $k \leq k$, thus $k \leq k'$. $\square$

Lemma 3.2.19 $\Vdash^{M^K}_{S^k} A \Longleftrightarrow \vDash^K_k A$.

Proof: $v(a, k) = T$ implies $\vdash_{S^k} a$ by construction, so $\vDash^K_k a$ implies $\Vdash^{M^K}_{S^k} a$. For the other direction, notice that all axiomatic rules in S^k are added by the procedure and that $\vdash_{S^k} a$ holds only if $[/a]$ was added to S^k (per Lemma 3.2.16 we can ignore the vacuous rules of S^k), thus $\vdash_{S^k} a$ implies $v(a, k) = T$ and so $\Vdash^{M^K}_{S^k} a$ implies $\vDash^K_k a$.

The proof proceeds as in Theorem 3.2.12, the only differences being that we substitute $\vDash^{KM}_S$ by $\vDash^K_k$, $\Vdash^M_S$ by $\Vdash^{M^K}_{S^k}$ and that, in the proof for $A \to B$, the equivalence $k \leq k'$ iff $S^k \subseteq_{M^k} S^{k'}$ holds due to Lemma 3.2.18 instead of by definition. $\square$

Corollary 3.2.20 $\Gamma \vDash^K A \Longleftrightarrow \Gamma \Vdash^{M^K} A$.

Lemma 3.2.21 $\Gamma \Vdash A$ implies $\Gamma \vDash^0_m A$.

Proof: Assume $\Gamma \Vdash A$. Then $\Gamma \Vdash^M A$ for any multibase M. Let K be any minimal propositional model. By Corollary 3.2.20 we have $\Gamma \vDash^K A \Longleftrightarrow \Gamma \Vdash^{M^K} A$. Since $\Gamma \Vdash^M A$ holds for all multibases M, we have $\Gamma \Vdash^{M^K} A$, thus $\Gamma \vDash^K A$. $\square$

From Lemmas 3.2.14 and 3.2.21 we immediately conclude:

Theorem 3.2.22 $\Gamma \vDash^0_m A \Longleftrightarrow \Gamma \Vdash A$.

The following also hold:

Theorem 3.2.23 $\Gamma \vDash_i^1 A \Longleftrightarrow \Gamma \vdash_i^1 A$.

Corollary 3.2.24 $\Gamma \vDash_m^0 A \Longleftrightarrow \Gamma \vdash_m^0 A$

Proof: See (VAN DALEN, 2013, pgs. 168-172). The proof for $\vdash_i^1$ can be transformed into a proof for $\vdash_m^0$ if we treat $\bot$ as a common 0-ary predicate and remove the steps for the treatment of first-order formulas that are not propositional formulas. □

From Theorem 3.2.22 and Corollary 3.2.24 follows soundness ($\Gamma \vdash_m^0 A \Rightarrow \Gamma \Vdash A$) and completeness ($\Gamma \Vdash A \Rightarrow \Gamma \vdash_m^0 A$) for standard multibase validity:

Theorem 3.2.25 (Soundness and Completeness) $\Gamma \Vdash A \Longleftrightarrow \Gamma \vdash_m^0 A$.

Which is what we wanted to prove.

In order to extend the result to focused validity, we define the following notion:

Definition 3.2.26 A model is *rooted* if there is a $k \in W$ such that $k \leq k'$ for all $k' \in W$. This k is called the model's *root*.

Which leads to the following results for focused validity:

Lemma 3.2.27 $\Gamma \Vdash_m^0 A$ iff $\Gamma \vDash^K A$ for all rooted minimal models K.

Proof: From Corollary 3.2.5 it follows that, if F is focused on S, for all $S' \in F$ we have $S \subseteq_F S'$. Since the procedures in Definitions and 3.2.11 and 3.2.17 preserve the structure of $\leq$ and $\subseteq$ (by construction in the first case and due to Lemma 3.2.18 in the second), by transforming a multibase focused on S we obtain a model with S as its root, and by transforming a model with root k and putting S^k at the beginning of the sequence we have a multibase focused on S^k. The result can then be obtained by restricting the reasoning used in Lemmas 3.2.14 and 3.2.21 to focused multibases and rooted models. □

The proof of the theorem here numbered as 3.2.24 and numbered as 6.3.10 in (VAN DALEN, 2013, pgs. 172) shows not only completeness for all models, but also for all *rooted* models:

Theorem 3.2.28 $\Gamma \vdash_i^1 A \Longleftrightarrow \Gamma \vDash^K A$ for all rooted first-order intuitionistic models K.

Corollary 3.2.29 $\Gamma \vdash_m^0 A \Longleftrightarrow \Gamma \vDash^K A$ for all rooted minimal propositional models K.

Proof: See once again the proof in (VAN DALEN, 2013, pgs. 168-172), especially his Lemma 6.3.9. By restricting the proof for $\vdash_i^1$ just as done to obtain Corollary 3.2.24 from Theorem 3.2.23 we obtain the result for $\vdash_m^0$. □

From this the desired result follows:

Theorem 3.2.30 (Focused Soundness and Completeness) $\Gamma \Vdash^0_m A \iff \Gamma \vdash^0_m A$.

Proof: The results follow immediately from Lemma 3.2.27 and Corollary 3.2.29. □

The results can be further improved:

Theorem 3.2.31 The class of all intuitionistic Kripke models that are finite trees with unique roots are sufficient to generate all counterexamples to intuitionistic validities.

Corollary 3.2.32 (Finite Soundness and Completeness) Let $\Gamma \Vdash_{fin} A$ hold if and only if $\Gamma \Vdash^M A$ holds for every finite multibase M and $\Gamma \Vdash^*_{fin} A$ hold if and only if $\Gamma \Vdash^F A$ holds for every finite focused multibase F. Then $\Gamma \Vdash_{fin} A \iff \Gamma \Vdash^*_{fin} A \iff \Gamma \vdash^0_m A$.

Proof: See Theorems 6.8 and 6.12 in (TROELSTRA; VAN DALEN, 1988), which may be adapted to show the same for minimal validity just like Theorem 3.2.23 was adapted to yield Corollary 3.2.24. □

Since we are mainly interested in focused multibases, from now on we deal only with them. It is generally straightforward to adapt each result so that it holds for multibases in general (except the results for generalized S-validity). All soundness and completeness results presented also hold for finitary versions of the corresponding multibases and focused multibases, but since the proofs usually involve a straightforward adaptation of the non-finitary proofs we omit them for the sake of simplicity.

We have adopted sequences to justify some philosophical intuitions, but one may clearly redesign focused multibases so they become *focused trees* or even *focused finite trees*, both with unique roots. As a bonus, the ordering mentioned before – no extension of a base precedes it in a sequence – is always present in uniquely rooted trees.

3.2.4 Classical and intuitionistic focused multibases

We now extend the results to classical and intuitionistic propositional logic. This is done in a separate section because the new framework allows different extensions with very different proof-theoretic properties, so we briefly discuss some of our choices.

There are at least four ways to obtain intuitionistic logic from minimal logic, all involving changing the semantics of $\bot$. The first two are shared between model-theoretic and proof-theoretic semantics, but the last two are unique to the latter:

Proposition 3.2.33 Intuitionistic multibase semantics can be obtained from minimal multibase semantics through any of the following modifications, for all S in all M:

1. $\nVdash^M_S \bot$;

2. $\Vdash_S^M \bot$ iff $\Vdash_S^M a$, for all atomic a;

3. S must be consistent (cf. Definition 2.5.9);

4. S must be explosive (cf. Definition 3.1.2).

The first two regard $\bot$ as a non-atomic formula and define it at the level of semantic clauses; the last consider it an atom and define it in terms of atomic derivability.

The first definition, commonly used in intuitionistic Kripke models, cannot be used in standard proof-theoretic validity or base-extension semantics because, if we always take into account all possible extensions of bases and allow every atom to be validated in some extension, it becomes possible to prove things such as $\vDash \neg\neg a$ for all atomic a (PIECHA; SANZ; SCHROEDER-HEISTER, 2015). This is not problematic in multibase semantics because we are allowed to limit the set of possible extensions so as to avoid this.

The second definition is not seen very often, but it is used in (SANDQVIST, 2015).

The third definition was presented in (NASCIMENTO; PEREIRA; PIMENTEL, 2023) to provide base-extension semantics for ecumenical logics. When requiring all bases to be consistent, we indirectly restrict the set of all possible extensions by only considering *consistent* extensions of bases. This makes it so that, for example, by including the following rule in a base:

$$\frac{a}{\bot}$$

We prevent its extensions from deriving a, since any such extension would be inconsistent. The end result is a successful implementation of the principle behind the first definition at the syntactic level, which works in base-extension semantics as well as in multibase semantics and generalized proof-theoretic validity.

Just like the third definition is the syntactic counterpart of the first, the fourth is the syntactic counterpart of the second. It is extensively used in the literature, and was also used in the completeness proof in (STAFFORD; NASCIMENTO, 2023).

Although the choice might look irrelevant at first glance, it has many important consequences. First of all, extension of the embedding in Definition 3.2.11 is not immediate if we use the second or fourth definitions. Some of the results for generalized S-validity that will be presented later also become less general if the first or second definitions are used, and there are even differences between the results one may obtain by using the third or the fourth. The final reason is that the definition one chooses impacts the possible ways of extending intuitionistic logic to classical logic.

The definition that works best for our purposes is the third one. Not only do the results for generalized S-validity become more interesting and the extension of Definition 3.2.11 becomes trivial, but the transition to classical logic also becomes smoother.

As such, we obtain intuitionistic multibases as follows:

Definition 3.2.34 A base is *intuitionistic* if it is consistent.

Definition 3.2.35 Let S be an intuitionistic base. An *intuitionistic focused multibase* for S is any focused multibase for S with only intuitionistic bases.

Definition 3.2.36 Let S be a intuitionistic base. $\Gamma \Vdash^i_S A$ iff $\Gamma \Vdash^F A$ holds for all intuitionistic multibases F focused on S. The relation $\Vdash^i_S$ will be called *intuitionistic generalized S-validity*.

Definition 3.2.37 $\Gamma \Vdash^0_i A$ iff $\Gamma \Vdash^i_S A$ for all intuitionistic S.

Lemma 3.2.38 $\Gamma \Vdash^0_i A$ iff $\Gamma \vDash^K$ for all rooted intuitionistic models K.

Proof: Since $\nvdash_S \bot$ holds for every S in any intuitionistic focused multibase, it is easy to see that Definition 3.2.11 yields a model such that, for all $k \in W$, $v(\bot, k) = \emptyset$, so it is a intuitionistic model (cf. Definition 2.3.3). Likewise, all multibases obtained from intuitionistic models through Definition 3.2.17 are easily seen to have only consistent bases, so they are intuitionistic bases. In both cases we can adapt the reasoning used to prove Lemma 3.2.27 to show that all models obtained are rooted and all multibases are focused (provided the S^k obtained from the root k is the first base of the sequence). Just like in the minimal case, the desired results follow if we apply the reasoning in prove Lemmas 3.2.14 and 3.2.21. □

The following is also a corollary of Theorem 3.2.28:

Corollary 3.2.39 $\Gamma \vdash^0_i A$ iff $\Gamma \vDash^K A$ for all intuitionistic rooted K;

Proof: Just like in the proof of Corollary 3.2.29, we restrict the proof of Theorem 3.2.28. The steps for the treatment of first-order formulas that are not propositional formulas are removed, but this time we retain the special treatment given to $\bot$. □

Now we may finally conclude:

Theorem 3.2.40 $\Gamma \Vdash^0_i A$ iff $\Gamma \vdash^0_i A$.

Proof: The result follows from Lemma 3.2.38 and Corollary 3.2.39. □

Just as in the previous case, there are many distinct ways to obtain classical multibases from intuitionistic multibases. We list four:

Proposition 3.2.41 Classical multibase semantics can be obtained from intuitionistic multibase semantics through any of the following modifications, imposed on all S and M:

1. If $\nVdash^M_S a$ and $S \subseteq_M S'$ then $\nVdash^M_{S'} a$.
2. $\Vdash^M_S a$ or $a \Vdash^M_S \bot$, for all atomic a;

3. If $\nvdash_S a$ and $S \subseteq_M S'$ then $\nvdash_{S'} a$.

4. For all S, either $\vdash_S a$ or $a \vdash_S \bot$.

The first two cannot be viewed as mere new clauses because they interfere with the semantics of all atomic formulas (instead of just $\bot$), so they are best viewed as properties expected of all multibases that will be used to define classical multibases.

Clearly, the third is the syntactic counterpart of the first, and the fourth of the second. Our preference must once again rest in the use of syntactic definitions due to their relation with generalized S-validity. Even though the third definition is closer to the restriction used on Definition 2.3.4, we choose the fourth because it is the only one defined without reference to the multibases containing it.

Definition 3.2.42 A intuitionistic base is *classical* if, for every a, either $\vdash_S a$ or $a \vdash_S \bot$.

Definition 3.2.43 Let S be a classical base. A *classical focused multibase* for S is a focused multibase for S with only classical bases.

Definition 3.2.44 Let S be a classical base. $\Gamma \Vdash^c_S A$ iff $\Gamma \Vdash^F A$ holds for all classical multibases F focused on S. The relation $\Vdash^c_S$ will be called *classical generalized S-validity*.

Definition 3.2.45 $\Gamma \Vdash^0_c A$ iff $\Gamma \Vdash^c_S A$ for all classical S.

We briefly mentioned after Definition 2.3.4 that, in the propositional case, it is sufficient to require preservation of non-truth of atoms to induce classical behavior. We now prove a similar result for multibases:

Lemma 3.2.46 For all A and all S in a classical focused multibase M, either $\Vdash^M_{S'} A$ for all $S \subseteq_M S'$ or $\nVdash^M_{S'} A$ for all $S \subseteq_M S'$.

Proof:

1. Atomic case: If $\vdash_S a$ for $a \neq \bot$, the atomic deduction showing this can be replicated in all extensions of S', so by Definition 3.2.3 we have $\Vdash^M_{S'} a$ for all $S \subseteq_M S'$. If $\nvdash_S a$, from the definition of classical bases it follows that $a \vdash_S \bot$. Since classical bases are also intuitionistic bases, every base in a classical focused multibase must be consistent. Now assume, for the sake of contradiction, that there is a extension S' of S such that $\Vdash^M_{S'} a$. Then we have $\vdash_{S'} a$. Since $S \subseteq S'$, we can reproduce the deduction showing $a \vdash_S \bot$ to show $a \vdash_{S'} \bot$. But then we may put the deduction showing $\vdash_{S'} a$ above each premise a on which $\bot$ depends in the deduction showing $a \vdash_{S'} \bot$ and produce a deduction showing $\vdash_{S'} \bot$, contradicting the consistency requirement. Therefore, $\nvdash_{S'} a$ for all $S \subseteq S'$, so $\nVdash^M_{S'} a$ for all $S \subseteq_M S'$.

   In the case of $\bot$ we immediately obtain $\nVdash^M_{S'} \bot$ for all $S \subseteq_M S'$ from the fact that all bases are consistent.

2. $A \wedge B$: If $\Vdash^M_{S'} A$ for all $S \subseteq_M S'$ and $\Vdash^M_{S'} B$ for all $S \subseteq_M S'$ we have $\Vdash^M_{S'} A \wedge B$ for all $S \subseteq_M S'$. If either $\nVdash^M_{S'} A$ for all $S \subseteq_M S'$ or $\nVdash^M_{S'} B$ for all $S \subseteq_M S'$, we have $\nVdash^M_{S'} A \wedge B$ for all $S \subseteq_M S'$.

3. $A \vee B$: If either $\Vdash^M_{S'} A$ for all $S \subseteq_M S'$ or $\Vdash^M_{S'} B$ for all $S \subseteq_M S'$, we have $\Vdash^M_{S'} A \vee B$ for all $S \subseteq_M S'$. If $\nVdash^M_{S'} A$ for all $S \subseteq_M S'$ and $\nVdash^M_{S'} B$ for all $S \subseteq_M S'$, we have $\nVdash^M_{S'} A \vee B$ for all $S \subseteq_M S'$.

4. $A \rightarrow B$: From transitivity of $\subseteq_M$ it follows that, for any $S \subseteq_M S''$, if $\Vdash^M_{S'} A$ for all $S \subseteq_M S'$ then $\Vdash^M_{S'''} A$ for all $S'' \subseteq_M S'''$, and if $\nVdash^M_{S'} A$ for all $S \subseteq_M S'$ then $\nVdash^M_{S'''} A$ for all $S'' \subseteq_M S'''$. Hence, if either $\nVdash^M_{S'} A$ for all $S \subseteq_M S'$ or $\Vdash^M_{S'} B$ for all $S \subseteq_M S'$, we have $\Vdash^M_{S'} A \rightarrow B$ for all $S \subseteq_M S'$. If both $\Vdash^M_{S'} A$ for all $S \subseteq_M S'$ and $\nVdash^M_{S'} B$ for all $S \subseteq_M S'$ we have $\nVdash^M_{S'} A \rightarrow B$ for all $S \subseteq_M S'$ □

We immediately get:

Corollary 3.2.47 If F is a classical focused multibase, $\Vdash^M_S A \vee \neg A$ for all $S \in F$.

Proof: If $\Vdash^M_S A$ the result is immediate. If $\nVdash^M_S A$ then, by Lemma 3.2.46, for all $S \subseteq S'$ we have $\nVdash^M_{S'} A$, vacuously satisfying the clause for $A \rightarrow \bot$ and yielding $\Vdash^M_S \neg A$, hence $\Vdash^M_S A \vee \neg A$. □

Which yields:

Theorem 3.2.48 $\Gamma \Vdash^0_c A$ iff $\Gamma \vdash^0_c A$.

Proof: Since classical bases are also intuitionistic bases, all classical focused multibases are also intuitionistic focused multibases, which together with Theorem 3.2.40, yields that $\Gamma \vdash^0_i A$ implies $\Gamma \Vdash^F A$ for all classical focused multibases F. Corollary 3.2.47 shows that $\Gamma \Vdash^M A \vee \neg A$ holds for every A in every classical focused multibase. But intuitionistic logic is obtained from classical logic by removing the excluded middle (MOSCHOVAKIS, 2022), so after revalidating all instances of it we are back in classical logic. □

3.3 Results for generalized S-validity

3.3.1 Reducibility of semantic values to derivability

In its conception, proof-theoretic validity aimed to reduce validity of logical connectives to validity in production bases (cf. Definition 2.5.6). S-validity of atoms would be established by considering their proofs in S, and S-validity of logical connectives by considering the rules of natural deduction together with proofs in S and its extensions (PRAWITZ, 2006)(SCHROEDER-HEISTER, 2006). Direct reducibility of logical validity to atomic derivability is one of the cornerstones of proof-theoretic validity.

S-validity is lost in general proof-theoretic validity and standard multibase validity. Bases are used only to establish what is valid in a particular proof-theoretic system or multibase, so their contribution to validity is indirect. This is no longer the case if we use focused validity and generalized S-validity. By once again focusing on bases we allow many properties characteristic of S-validity to be proved also for its generalized version. Not only so, but it is possible to argue (as we would like to) that generalized S-validity is an effective implementation of the idea behind Prawitz's and Dummett's definitions.

We start our discussions by proving an important theorem:

Theorem 3.3.1 $\{a^1, \ldots, a^n\} \Vdash_S b$ iff $\{a^1, \ldots, a^n\} \vdash_S b$.

Proof:

($\Rightarrow$): Assume $\{a^1, \ldots, a^n\} \Vdash_S b$. Then $\{a^1, \ldots, a^n\} \Vdash^F b$ for every F focused on S, so for every such F we have $\{a^1, \ldots, a^n\} \Vdash^F_{S'} b$ for every $S' \in F$. Let S'' be the base obtained by adding to S one rule $[/a^m]$ for each $1 \leq m \leq n$, and F^* the focused multibase $\langle S, S'' \rangle$. Clearly, $\vdash_{S''} a^m$ holds for every $a^m \in \{a^1, \ldots, a^n\}$, so by putting $F = F^*$ and $S' = S''$ to obtain $\{a^1, \ldots, a^n\} \Vdash^{F^*}_{S''} b$ we have $\Vdash_{S''} b$ by clauses 5 and 6 of Definition 3.2.3, hence $\vdash_{S''} b$. As such, there must be a deduction Π concluding b which depends on no premises and uses only the rules of S''.

If Π does not use any of the rules added to S to obtain S'', it is already a deduction in S, which shows $\vdash_S b$ and then $\{a^1, \ldots, a^n\} \vdash_S b$ (cf. Definitions 2.5.8 and 2.2.5).

If Π does use some of the rules added to S, let Π' be the deduction obtained from Π by substituting all application of rules $[/a_{S''}]$ such that $[/a_{S''}] \in S''$ but $[/a_{S''}] \notin S$ by assumptions with shape $a_{S''}$. Represented in tree form, the procedure is as follows:

$$\begin{array}{c} \overline{a^1_{S''}} \quad \ldots \quad \overline{a^m_{S''}} \\ \Pi \\ b \end{array} \quad \text{is transformed into} \quad \begin{array}{c} a^1_{S''} \quad \ldots \quad a^m_{S''} \\ \Pi' \\ b \end{array}$$

Notice that, since all rules in S'' but not in S were removed from Π, Π' only contains rules in S, so Π' is a deduction showing $\{a^1_{S''}, \ldots, a^m_{S''}\} \vdash_S b$. But if a rule $[/a]$ was added to S in the construction of S'' then $[/a] \in \{a^1, \ldots, a^n\}$, hence $\{a^1_{S''}, \ldots, a^m_{S''}\} \subseteq \{a^1, \ldots, a^n\}$, thus $\{a^1, \ldots, a^n\} \vdash_S b$.

($\Leftarrow$): Assume $\{a^1, \ldots, a^n\} \vdash_S b$, which is shown by a deduction Π. Let F be any multibase focused on S. Let S'' be any base with $S'' \in F$ and $\Vdash^F_{S''} \{a^1, \ldots, a^n\}$. Then for every a^m ($1 \leq m \leq n$) there must be a deduction Π^m concluding a^m and depending on no formulas. Since Π is a deduction in S and $S \subseteq_F S''$, Π is also a deduction in S''. We compose the deductions Π^m with Π to obtain a deduction of b depending on no formulas:

$$
\begin{array}{ccc}
\Pi^1 & & \Pi^n \\
a^1 & ... & a^n \\
& \Pi & \\
& b &
\end{array}
$$

This shows $\vdash^F_{S''} b$, so $\Vdash^F_{S''} b$. Since this is also a deduction on every extension of S'', we conclude $\Vdash^F_{S'''} b$ for all $S'' \subseteq_F S'''$, so $\{a^1, \, ... \, , a^n\} \Vdash^F_{S''} b$ and, since S'' is arbitrary, $\{a^1, \, ... \, , a^n\} \Vdash^F b$. But F is also a arbitrary multibase focused on S, hence $\{a^1, \, ... \, , a^n\} \Vdash_S b$. □

This theorem shows that generalized S-validity for atoms is reducible to atomic derivability in S. Since our definitions of validity are recursive, generalized S-validity of all connectives is ultimately reducible to derivability in S. Semantic values are entirely determined by derivability, and so is semantic consequence. This is essentially what was intended by Prawitz and Dummett (PIECHA; SCHROEDER-HEISTER, 2016, pg. 49)(PRAWITZ, 1971)(DUMMETT, 1991), so generalized S-validity can be considered an effective implementation of the ideas originally behind proof-theoretic validity.

If we consider specifically the case in which the context Γ of $\Gamma \Vdash A$ is empty, some additional results can be obtained. First we prove some lemmata:

Lemma 3.3.2 (Monotonicity) For any focused multibase F and any $S \in F$, $\Vdash^F_S A$ and $S \subseteq_F S'$ implies $\Vdash^F_{S'} A$.

Proof:

1. Atomic case: immediate from the fact that if there is a deduction Π showing $\vdash_S a$ then it is also a deduction showing $\vdash_{S'} a$, since S' has the rules of S.

2. $(A \wedge B)$: Assume $\Vdash^F_S A \wedge B$. Then $\Vdash^F_S A$ and $\Vdash^F_S B$. Induction hypothesis: if $S \subseteq_F S'$ then $\Vdash^F_{S'} A$ and $\Vdash^F_{S'} B$. Hence $\Vdash^F_{S'} A \wedge B$.

3. $(A \vee B)$: Assume $\Vdash^F_S A \vee B$. Then $\Vdash^F_S A$ or $\Vdash^F_S B$. Induction hypothesis: if $S \subseteq_F S'$, then $\Vdash^F_S A$ implies $\Vdash^F_{S'} A$ and $\Vdash^F_S B$ implies $\Vdash^F_{S'} B$. Since either $\Vdash^F_S A$ or $\Vdash^F_S B$, either $\Vdash^F_{S'} A$ or $\Vdash^F_{S'} B$, hence $\Vdash^F_{S'} A \vee B$.

4. $(A \to B)$: Assume $\Vdash^F_S A \to B$. Then $A \Vdash^F_S B$, so for every $S \subseteq_F S'$ it holds that $\Vdash^F_{S'} A$ implies $\Vdash^F_{S'} B$. By transitivity of $\subseteq_F$ we have that $S' \subseteq S''$ implies $S \subseteq S''$, hence for all extensions S'' of S' it holds that $\Vdash^F_{S''} A$ implies $\Vdash^F_{S''} B$, which yields $A \Vdash^F_{S'} B$ and then $\Vdash^F_{S'} A \to B$. □

Lemma 3.3.3 (Isomorphism) Let M and M' be multibases. Let $Q = \langle S_1, S_2, ... \rangle$ be a subsequence of M such that $S_n \subseteq_M S$ implies $S \in Q$ for all $(n \geq 1)$, and $Q' = \langle S^{'}_1, S^{'}_2, ... \rangle$ a subsequence of M' such that $S^{'}_n \subseteq_{M'} S'$ implies $S' \in Q'$ for all $(n \geq 1)$. If it holds that $\vdash_{S_n} a$ if and only if $\vdash_{S^{'}_n} a$ and $S_n \subseteq_M S_m$ if and only if $S^{'}_n \subseteq_{M'} S^{'}_m$ for all $(n \geq 1)$ and $(m \geq 1)$, then $\Vdash^M_{S_n} A$ if and only if $\Vdash^{M'}_{S^{'}_n} A$, and also $\Gamma \Vdash^M_{S_n} A$ if and only if $\Gamma \Vdash^M_{S^{'}_n} A$.

Proof: We start by proving $\Vdash^{M}_{S_n} A$ if and only if $\Vdash^{M'}_{S'_n} A$ for arbitrary n.

1. Atomic case: immediate from the fact that $\vdash_{S_n} a$ if and only if $\vdash_{S'_n} a$ for every a;

2. $(A \wedge B)$: $\Vdash^{M}_{S_n} A \wedge B$. Then $\Vdash^{M}_{S_n} A$ and $\Vdash^{M}_{S_n} B$. Induction hypothesis: $\Vdash^{M'}_{S'_n} A$ and $\Vdash^{M'}_{S_n} B$. Then $\Vdash^{M'}_{S'_n} A \wedge B$. The converse can be proved in a similar fashion.

3. $(A \vee B)$: $\Vdash^{M}_{S_n} A \vee B$. Then $\Vdash^{M}_{S_n} A$ or $\Vdash^{M}_{S_n} B$. Induction hypothesis: $\Vdash^{M'}_{S'_n} A$ or $\Vdash^{M'}_{S_n} B$. If $\Vdash^{M'}_{S'_n} A$ then $\Vdash^{M'}_{S'_n} A \vee B$, and if $\Vdash^{M'}_{S'_n} B$ then $\Vdash^{M'}_{S'_n} A \vee B$, so in any case $\Vdash^{M'}_{S'_n} A \vee B$. The converse can be proved in a similar fashion.

4. $(A \to B)$: Assume $\Vdash^{M}_{S_n} A \to B$. Then $A \Vdash^{M}_{S_n} B$, so for every $S_n \subseteq_M S$ it holds that $\Vdash^{M}_{S} A$ implies $\Vdash^{M}_{S} B$. Since $S_n \subseteq S$ implies $S \in Q$ we have that all such bases S are $S_m \in Q$, so for all $S_n \subseteq_M S_m$ it holds that $\Vdash^{M}_{S_m} A$ implies $\Vdash^{M}_{S_m} B$. Induction hypothesis: if $S_n \subseteq_M S_m$ then $\Vdash^{M'}_{S'_m} A$ implies $\Vdash^{M'}_{S'_m} B$. But $S_n \subseteq_M S_m$ if and only if $S^{'}_n \subseteq_{M'} S^{'}_m$ and $S_n \subseteq_{M'} S'$ implies S' is some $S^{'}_m \in Q'$, hence $\Vdash^{M'}_{S'_m} A$ implies $\Vdash^{M'}_{S'_m} B$ for every $S^{'}_n \subseteq_{M'} S^{'}_m$ and so for all $S^{'}_n \subseteq_{M'} S'$, hence $A \Vdash^{M'}_{S'_n} B$, whence $\Vdash^{M'}_{S'_n} A \to B$. The converse can be proved in a similar fashion.

   The previous steps have shown $\Vdash^{M}_{S_n} C$ if and only if $\Vdash^{M'}_{S'_n} C$ for all C. To finish the proof, assume $\Gamma \Vdash^{M}_{S_n} A$. Then for every $S_n \subseteq_M S$ it holds that $\Vdash^{M}_{S} B$ for every $B \in \Gamma$ implies $\Vdash^{M}_{S} A$, and again for all $S_n \subseteq_M S_m$ we have $\Vdash^{M}_{S_m} B$ for all $B \in \Gamma$ implies $\Vdash^{M}_{S_m} A$. For all $S^{'}_n \subseteq_{M'} S^{'}_m$ we have $\Vdash^{M}_{S_m} C$ if and only if $\Vdash^{M'}_{S'_m} C$ for all C, so $\Vdash^{M'}_{S'_m} B$ for every $B \in \Gamma$ implies $\Vdash^{M'}_{S'_m} A$. But for every $S^{'}_n \subseteq_{M'} S'$ we have that S' is some $S^{'}_m \in Q'$, so this holds for all $S^{'}_{n'} \subseteq_{M'} S'$, hence $\Gamma \Vdash^{M'}_{S'_n} A$. The converse can be proved in a similar fashion. □

Then we prove the desired results:

Theorem 3.3.4 The following equivalences hold:

1. $\Vdash_S a \iff \vdash_S a$;
2. $\Vdash_S A \to B \iff A \Vdash_S B$;
3. $\Vdash_S A \wedge B \iff \Vdash_S A$ and $\Vdash_S B$;
4. $\Vdash_S A \vee B \iff \Vdash_S A$ or $\Vdash_S B$.

Proof:

1. Atomic case: immediate from Theorem 3.3.1.

2. $A \to B$: Assume $\Vdash_S A \to B$. Let F be any multibase focused on S. Then $\Vdash^{F} A \to B$ holds, and so $\Vdash^{F}_{S'} A \to B$ for every $S' \in F$. From this we conclude $A \Vdash^{F}_{S'} B$ for every such S', thus $A \Vdash^{F} B$. But F is an arbitrary multibase focused on S, so $A \Vdash_S B$. For the

other direction, assume $A \Vdash_S B$, and let F be any multibase focused on S. Then $A \Vdash^F_{S'} B$ holds for every $S' \in F$, hence $\Vdash^F_{S'} A \to B$ for every such S' and so $\Vdash^F A \to B$. Since F is again arbitrary, we conclude $\Vdash_S A \to B$.

3. $A \wedge B$: Assume $\Vdash_S A \wedge B$, and let F be any multibase focused on S. Then $\Vdash^F A \wedge B$ holds and thus $\Vdash^F_{S'} A \wedge B$ for any $S' \in F$. Hence $\Vdash^F_{S'} A$ and $\Vdash^F_{S'} B$ for every $S' \in F$, so both $\Vdash^F A$ and $\Vdash^F B$ and, since F is arbitrary, $\Vdash_S A$ and $\Vdash_S B$. For the other direction, assume $\Vdash_S A$ and $\Vdash_S B$, and let F be an arbitrary multibase focused on S. Then $\Vdash^F A$ and $\Vdash^F B$, hence $\Vdash^F_{S'} A$ and $\Vdash^F_{S'} B$ for all $S' \in F$, hence $\Vdash^F_{S'} A \wedge B$ for all S', hence $\Vdash^F A \wedge B$ for this arbitrary F, whence $\Vdash_S A \wedge B$.

4. $A \vee B$: The proof of this case is much more involved. Our strategy for proving the left-to-right direction consists in picking one focused multibase yielding a counterexample to the generalized S-validity of A, another yielding a counterexample to the generalized S-validity of B, and then melding them together to construct a single focused multibase which is a counterexample to the S-validity of $A \vee B$, which is then used to conclude the desired result by contraposition. The converse is much simpler to prove.

   Assume $\nVdash_S A$ and $\nVdash_S B$. Then there must be a multibase F focused on S such that $\nVdash^F A$ and a multibase F' focused on S such that $\nVdash^{F'} B$. For any $S^k \in F$, let S^k_F be the system obtained by adding a vacuous rule $[a^{k'}_F / a^{k'}_F]$ to S^k for every $S^{k'} \subseteq_F S^k$. Likewise, for any $S^j \in F'$, let $S^j_{F'}$ be the base obtained by adding a vacuous rule $[a^{j'}_{F'} / a^{j'}_{F'}]$ to $S^j \in F'$ for every $S^{j'} \subseteq_{F'} S^j$. Assume that none of the new vacuous rules were originally in some base of the multibase and, if the language does not contain a sufficient amount of atoms (e.g. one of the bases contain all possible vacuous rules), enrich the language so that new rules become available[55]. Assume also that the rules have been chosen so that no rule added in the procedure for F is added in the procedure for F' and vice-versa.

   Let F'' be the focused multibase starting with S and containing precisely the S^k_F and S^j_F previously obtained. The same reasoning used in Lemma 3.2.18 can be used here to show that $S^k_F \subseteq_{F''} S^{k'}_F$ if and only if $S^k \subseteq_F S^{k'}$ and $S^j_{F'} \subseteq_{F''} S^{j'}_{F'}$ if and only if $S^j \subseteq_{F'} S^{j'}$. Since only vacuous rules were added, derivability of atoms remains the same per Lemma 3.2.16. Notice that, for any new atom b added to the language to produce vacuous rules, we have ($\nvdash_{S^k} b$), ($\nvdash_{S^k_F} b$), ($\nvdash_{S^j} b$) and ($\nvdash_{S^j_{F'}} b$) since b does not occur in S^k or S^j and only possibly occurs in S^k_F or $S^j_{F'}$ as premise and conclusion of vacuous rules. Since $S^k_F \subseteq_{F''} S^{k'}_F$ if and only if $S^k \subseteq_F S^{k'}$, for every C we have $\Vdash^{F''}_{S^k_F} C$ if and only if $\Vdash^F_{S^k} C$

[55] In order to prove the result for a particular language, we can simply start by considering multibases for a reduced language (e.g. the language obtained by numbering every 0-ary constant of the target language and deleting every predicate that receives a even number) and then enrich the reduced language in such a way that it becomes precisely the target language.

per Lemma 3.3.3. The same applies to the S^j, so $\Vdash^{F''}_{S^j_{F'}} C$ if and only if $\Vdash^{F'}_{S^j} C$.

Now assume $\Vdash^{F''}_{S} A \vee B$. Then either $\Vdash^{F''}_{S} A$ or $\Vdash^{F''}_{S} B$. In the first case we get $\Vdash^{F''}_{S'} A$ for all $S' \in F''$ by Lemma 3.3.2. But since $S \subseteq_{F''} S^k_F$ for every k it is also the case that $\Vdash^{F''}_{S^k_F} A$ for every k and thus $\Vdash^{F}_{S^k} A$ for every $S^k \in F$, yielding $\Vdash^{F} A$ and contradicting the assumption that $\nVdash^{F} A$. Likewise, if $\Vdash^{F''}_{S} B$ then $\Vdash^{F''}_{S^j_{F'}} B$ for every j and thus $\Vdash^{F'}_{S^j} B$ for every $S^j \in F$, yielding $\Vdash^{F'} B$ and contradicting the assumption that $\nVdash^{F'} B$. We finally conclude $\nVdash^{F''}_{S} A \vee B$, thus $\nVdash^{F''} A \vee B$ and $\nVdash_S A \vee B$.

For the other direction, assume that either $\Vdash_S A$ or $\Vdash_S B$. If $\Vdash_S A$ we have $\Vdash^F A$ for all F focused on S, thus for arbitrary F we have $\Vdash^F_{S'} A$ for all $S' \in F$, yielding $\Vdash^F_{S'} A \vee B$ for all $S' \in F$ and then $\Vdash^F A \vee B$, thus $\Vdash_S A \vee B$. A simular argument proves that $\Vdash_S B$ implies $\Vdash_S A \vee B$, so in any case we have $\Vdash_S A \vee B$. □

Theorem 3.3.5 $A \Vdash_S B$ implies ($\Vdash_S A \Rightarrow \Vdash_S B$).

Proof: Assume $\Vdash_S A$ and $A \Vdash_S B$. Then in every multibase F focused on S we have both $\Vdash^F A$ and $\Vdash^F A \to B$, hence $\Vdash^F_{S'} A$ and $\Vdash^F_{S'} A \to B$ for all $S' \in F$. Since $\Vdash^F_{S'} A \to B$ we also have $A \Vdash^F_{S'} B$, which combined with $\Vdash^F_{S'} A$ yields $\Vdash^F_{S'} B$; since S' is arbitrary we have $\Vdash^F B$, and since F was arbitrary we have $\Vdash_S B$. □

Theorem 3.3.6 It does not hold that $(\Vdash_S A \Rightarrow \Vdash_S B)$ implies $A \Vdash_S B$.

Proof: Let $A = a$ and $B = b$ for any two atoms with $a \neq b$. Consider the focused multibase $F = \langle S = \{\emptyset\}, S' = \{[/a]\}\rangle$. Since $\nvdash^F_S a$ we have that $(\Vdash_S a \Rightarrow \Vdash_S b)$ is vacuously satisfied. But since $\vdash^F_{S'} a$ and $\nvdash^F_{S'} b$ we have $\Vdash^F_{S'} a$ and $\nVdash^F_{S'} b$, thus $a \nVdash^F_{S'} b$, hence $a \nVdash^F b$ and so $a \nVdash_S b$, so the consequent is not satisfied. □

In the next section it will become clear that there are good reasons for the failure of this principle. In any case, we should not be misled by the previous results: generalized S-validity and semantic consequence is still defined by considering validity in focused multibases, not by recourse to the clauses in Theorem 3.3.4.

The reason why generalized S-validity works (and traditional S-validity doesn't) is that the structure of focused multibases enables the production of many new counterexamples to validities. In fact, as seen before, if a Kripke model with root k is a counterexample to validity of some formula, the corresponding focused multibase will also be a counterexample to validity of the same formula. When we look at Theorem 3.3.4 and Definition 3.1.1 things seem more or less the same, but as soon as we try to find a counterexample for something we notice that the underlying structure is much richer.

3.3.2 Binding by bases, Export and Import

Many results for proof-theoretic validity, base-extension semantics and similar notions depend on the kind of atomic bases being used. Results are often restricted to specific bases; some results for production bases do not hold when we consider standard or higher-order ones[56]. S-validity is usually bound in some way by the rules of S, so by allowing new types of rules one may also be allowing different kinds of binding.

Theorem 3.3.1 already suggests that results of this kind are present in generalized S-validity. We examine this topic in further detail:

Theorem 3.3.7 Let S be any base containing a rule with the following shape:

$$\frac{a^1 \quad \ldots \quad a^n}{b}$$

Then $\Vdash_S (a^1 \wedge \ \ldots \ \wedge a^n) \to b$

Proof: A single application of the rule already shows $\{a^1, \ \ldots \ , a^n\} \vdash_S b$, hence by Theorem 3.3.1 we have $\{a^1, \ \ldots \ , a^n\} \Vdash_S b$ and so $\{a^1, \ \ldots \ , a^n\} \Vdash^F_{S'} b$ for any S' in any F focused on S. If on any such S' we have that $\Vdash_{S'} a^1 \wedge \ldots \wedge a^n$ holds, iterated applications of the clause for conjunction yields $\Vdash^F_{S'} \{a^1, \ \ldots \ , a^n\}$, hence $\Vdash^F_{S'} b$, hence $a^1 \wedge \ \ldots \ \wedge a^n \Vdash^F_{S'} b$, whence $\Vdash^F_{S'} (a^1 \wedge \ \ldots \ \wedge a^n) \to b$, which by arbitrariness of S' and F yields $\Vdash_S (a^1 \wedge \ \ldots \ \wedge a^n) \to b$. $\square$

Production rules thus bind generalized S-validity for the S in which they occur. Curiously, this is not the case for rules discharging formulas or other rules, so the result does not generalize:

Theorem 3.3.8 Let S contain one of the following rules, and no other:

$$\begin{array}{c} [a] \\ \vdots \\ \dfrac{b}{c} \end{array} \qquad\qquad \begin{array}{c} [/a] \\ \vdots \\ \dfrac{b}{c} \end{array}$$

Then $\nVdash_S (a \to b) \to c$.

Proof: Assume S contains only the rule on the left, and let F be a multibase containing only S. Notice that, since there are no rules with a or b as their conclusions, there cannot be a proof of a or of b in S. Since there is no rule with conclusion b, it is also the case that an

[56] For some examples, see the completeness and incompleteness results for the disjunctionless fragment of the language in (SANDQVIST, 2009)(SANDQVIST, 2009)(PIECHA; SANZ; SCHROEDER-HEISTER, 2015).

application of the rule of S always depends on the assumption b, so derivations of c always depend on either c or b. Then we have $\nvdash_S a$, $\nvdash_S b$ and $\nvdash_S c$. But since $\nvdash_S a$ and the only extension of S in the multibase is itself, $a \Vdash^F_S b$ holds vacuously, hence $\Vdash^F_S a \to b$. But since $\nvdash_S c$ we have $\nVdash^F_S c$, hence $a \to b \nVdash^F_S c$ and thus $\nVdash^F_S (a \to b) \to c$, hence $\nVdash^F (a \to b) \to c$, whence $\nVdash_S (a \to b) \to c$. A similar argument establishes the result also for a system containing only the rule on the right. □

Production rules bind generalized S-validity more strongly than rules discharging hypothesis, or even discharging other rules. This is not to say that such rules do not bind extensions at all: if a base S has the rule on the left, whenever some extension $S \subseteq S'$ has a deduction showing $a \vdash_{S'} b$ a single application of the rule yields a deduction showing $\vdash_{S'} c$. The same holds for the rule on the right in the presence of a deduction showing $[/a] \vdash_{S'} b$. This may be particularly important if one considers focused multibases on which particular restrictions are imposed – as shown by the case of saturated focused multibases, since both standard and higher-order rules strongly bind the original notion of S-validity (PIECHA; SANZ; SCHROEDER-HEISTER, 2015).

Those results are related to two general properties studied in the literature of proof-theoretic semantics:

Definition 3.3.9 (Export) A proof-theoretic semantics satisfies *Export* iff, for every S, there is a set of $\vee$-free formulas S^* such that $\Gamma \Vdash_S A \iff \Gamma, S^* \Vdash A$ holds.

Definition 3.3.10 (Import) A proof-theoretic semantics satisfies *Import* iff, for every S, every $\vee$-free Γ and every A, there is a base $S + \Gamma$ such that $\Gamma \Vdash_S A \iff \Vdash_{S+\Gamma} A$.

Although stated very generally, such properties are often studied alongside something we call the *standard mapping* of rules into formulas:

Definition 3.3.11 For any set $\Gamma = \{c^1, \dots, c^n\}$ of atoms or set $\Delta = \{[/c^1], \dots, [/c^n]\}$ of level 0 atomic higher-order rules, $\bigwedge \Gamma$ and $\bigwedge \Delta$ are defined as $c^1 \wedge \dots \wedge c^n$.

Definition 3.3.12 The *standard mapping* is defined as follows:

1. Every axiomatic rule and higher-order level 0 rule with shape $[/a]$ is mapped to the atom a;

2. Every non-axiomatic atomic rule with shape $[\{\Gamma^1\} \Rightarrow a^1, \dots, \{\Gamma^n\} \Rightarrow a^n/b]$ is mapped to the formula $((\bigwedge \Gamma^1 \to a^1) \wedge \dots \wedge (\bigwedge \Gamma^n \to a^n)) \to b$;

3. Every level 1 higher-order rule with shape $[\{\Delta^1\} \Rightarrow a^1, \dots, \{\Delta^n\} \Rightarrow a^n/b]$ is mapped to the formula $((\bigwedge \Delta^1 \to a^1) \wedge \dots \wedge (\bigwedge \Delta^n \to a^n)) \to b$;

4. Every higher-order rule with shape $[\{\Delta^1\} \Rightarrow a^1,\ ...\ ,\{\Delta^n\} \Rightarrow a^n/b]$ and level greater than 1 is mapped to the formula $((\bigwedge \Delta^1_* \rightarrow a^1) \wedge\ ...\ \wedge (\bigwedge \Delta^n_* \rightarrow a^n)) \rightarrow b$, in which each set Δ^m_* $(1 \leq m \leq n)$ is defined as the conjunction of all formulas to which the rules in Δ^m are mapped.

5. If a rule R is mapped to a formula F, F is also mapped[57] to R.

Definition 3.3.13 Let S be either a base or a higher-order base. Then S^* is the set of all formulas to which the rules of S are mapped.

From the structure of the mapping it follows that production rules do not allow iteration of implications on the antecedent of the formula, rules discharging formulas allow at most one iteration per premise, and higher-order rules with level n allow at most $n - 1$ iterations per premise. This makes it so that the mapping for higher-order rules is much more general than the mapping for for standard rules, so Import can only reasonably be expected to hold if we use higher-order bases.

It is shown in (PIECHA; SCHROEDER-HEISTER, 2019) that if we have the results proved in Theorem 3.3.4 and Theorem 3.3.5, the converse of Theorem 3.3.5 and either Import or Export, incompleteness of the semantics ensues. Those very general results are blocked by Theorem 3.3.6, which shows that the converse of Theorem 3.3.5 does not hold, thus invalidating important steps of the proof (e.g. the last step of the proof of Lemma 2.1 in (PIECHA; SCHROEDER-HEISTER, 2019)).

We are still left with the task of investigating whether Import and Export hold for the standard mapping. Some negative results are immediately obtainable from our preliminary theorems:

Theorem 3.3.14 Export does not hold in general if the standard mapping is used.

Proof: Immediate from the proof Theorem 3.3.8. □

Theorem 3.3.15 Import does not hold if the standard mapping is used.

Proof: Consider the valid consequence $((a \rightarrow b) \rightarrow c) \Vdash ((a \rightarrow b) \rightarrow c)$. If we consider only standard bases, the rule mapped to those formulas is $[\{a\} \Rightarrow b/c]$, so let $F = \langle S = \{[\{a\} \Rightarrow b/c]\}\rangle$. By Theorem 3.3.8 it follows that $\nVdash_S (a \rightarrow b) \rightarrow c)$. The same can be proved for the higher-order bases through the rule $[\{[/a]\} \Rightarrow b/c]$. □

We say that Export does not hold *in general* but Import does not hold *tout court* because the former is defined by existentially and the latter by universally quantifying over bases,

[57] Notice that, since standard rules and higher-order rules might be mapped to the same formula, the mapping from formulas to rules is not unique.

so the provided counterexamples show failure of Import in its entirety and failure of Export specifically in bases with rules discharging either formulas or other rules.

Fortunately, we can also obtain positive results if we only admit production bases in our semantics:

Definition 3.3.16 A *production multibase* is a focused multibase containing only production rules.

Lemma 3.3.17 $\Gamma, R^* \Vdash_S A$ if and only if $\Gamma \Vdash_{S \cup R} A$, provided only production multibases are admitted in the semantics.

Proof: We start each step of the proof by considering only rules with non-empty premises, but the proof for atomic axioms is provided immediately after.

1. Assume $(\Gamma, R^* \Vdash_S A)$. By Theorem 3.3.7 and Definition 3.3.12 we have $\Vdash_{S \cup R} R^*$, thus $\Vdash^F_{S^1} R^*$ for every multibase F focused on $S \cup R$ and every $S^1 \in F$. Now pick any focused multibase $F = \langle S \cup R, ... \rangle$, and let $F^1 = \langle S, S \cup R, ... \rangle$. From our initial assumption it follows that $\Gamma, R^* \Vdash^{F^1} A$, hence $\Gamma, R^* \Vdash^{F^1}_{S^2} A$ for all $S^2 \in F^1$. Since S does not extend any base in F^1 other than itself we can remove it to reobtain F and conclude $\Gamma, R^* \Vdash^F A$ per Lemma 3.3.3, thus $\Gamma, R^* \Vdash^F_{S^2} A$ for all $S^2 \in F$. Now assume there is a $S^3 \in F$ such that $\Vdash^F_{S^3} \Gamma$. By putting $S^1 = S^2 = S^3$ we have $\Vdash^F_{S^3} \Gamma, R^*$ and $\Gamma, R^* \Vdash^{F^1}_{S^3} A$, hence $\Vdash^F_{S^3} A$ and so $\Vdash^F_{S^4} A$ for all $S^3 \subseteq S^4$ by monotonicity, which shows $\Gamma \Vdash^F_{S^3} A$, hence $\Gamma \Vdash_{S \cup R} A$ follows from the arbitrariness of S^3 and F.

   Theorem 3.3.7 is not required to prove the result for atomic axioms, since $[/a] \in S$ obviously implies $\Vdash_S a$. The proof is otherwise identical.

2. Assume $\Gamma \Vdash_{S \cup R} A$. Let $R = [a^1, \, ... a^n / b]$. Then we have $R^* = (a^1 \wedge \, ... \wedge \, a^n) \to b$. Let F be any multibase focused on S, and assume $\Vdash^F_{S^1} \Gamma \cup R^*$ for some $S^1 \in F$. Assign a vacuous rule $[a^{S^2} / a^{S^2}]$ to every $S^2 \in F$, and assume that the atoms of each rule do not occur on rules of any base in F (we once again enrich the language if necessary). Let S^2_v be the base obtained from S^2 by adding the rule $[a^{S^3} / a^{S^3}]$ whenever $S^3 \subseteq_F S^2$. Now let F^1 be the multibase focused on $S \cup R$ such that, for bases distinct from $S \cup R$, for every $S^2 \in F$, we have $(S^2_v \cup R) \in F^1$ if and only if $S^1 \subseteq_F S^2$. From our assumptions it follows that $\Gamma \Vdash^{F^1} A$ and, once again, since validity in a base depends only on validity in its extensions which occur in the multibase per isomorphism, by omitting the $S \cup R$, possibly reordering the F^1 and putting $S^1_v \cup R$ at the beginning of the sequence we can apply Lemma 3.3.3 to get a multibase F^2 focused on $S^1_v \cup R$ such that $\Gamma \Vdash^{F^2}_{S^2_v \cup R} A$ for all $(S^2_v \cup R) \in F^2$.

   Let Π be any deduction showing $\vdash_{S^2_v \cup R} c$ for some c and some $(S^2_v \cup R) \in F^2$. Remove all vacuous rules from it through Lemma 3.2.16 to obtain a deduction Π'. If Π' does not use

R, it is already a deduction showing $\vdash_{S^2} c$. If it does use R, then there is some application of R in Π' which does not stand below any other[58], so the deduction has the following shape:

$$\dfrac{\begin{matrix}\Pi^1 & & \Pi^n \\ a^i & (...) & a^n\end{matrix}}{\begin{matrix} b \\ \Pi^j \\ c \end{matrix}}\, I$$

Since the rules of production bases are incapable of discharging formulas, the deductions Π^1 through Π^n cannot contain any open assumptions. Since we picked the topmost application of R, those deductions also do not contain applications of R, so they are deductions showing $\vdash_{S^2} a^m$ for every $1 \leq m \leq$, hence $\Vdash^F_{S^2} \{a^1 , ..., a^n\}$. But since $\Vdash^F_{S^1} R^*$ and $S^1 \subseteq S^2$ we have $\Vdash^F_{S^2} (a^1 \wedge \ ... \ \wedge a^n) \to b$ by monotonicity, hence $(a^1 \wedge \ ... \ \wedge a^n) \Vdash^F_{S^2} b$, hence $\{a^1 , ..., a^n\} \Vdash^F_{S^2} b$, whence $\Vdash^F_{S^2} b$, so there must be a deduction Π^k showing $\vdash_{S^2} b$. We can then transform the deduction Π' showing $\vdash_{S^2 \cup R} c$ into a deduction Π'' showing $\vdash_{S^2} c$ by removing every application of R in Π through reiterated use of the following transformation:

$$\dfrac{\begin{matrix}\Pi^1_* & & \Pi^n_* \\ a^i & (...) & a^n\end{matrix}}{\begin{matrix} b \\ \Pi^j_* \\ c \end{matrix}}\, I \qquad \text{is transformed into} \qquad \begin{matrix}\Pi^k \\ b \\ \Pi^j_* \\ c\end{matrix}$$

So Π'' is a deduction showing $\vdash_{S^2} c$, from which we conclude that $\vdash_{S^2_v \cup R} c$ implies $\vdash_{S^2} c$. Since the converse immediately follows from the fact that $S^2 \subseteq (S^2_v \cup R)$, we conclude that $\vdash_{S^2} c$ holds if and only if $\vdash_{S^2_v \cup R} c$ holds. Since by construction it holds that $(S^4_v \cup R) \subseteq_{F^2} (S^5_v \cup R)$ if and only if $S^4 \subseteq_F S^5$ for any extensions S^4 and S^5 of S^1 (as this is structure is induced by the vacuous rules added to bases) and, additionally, $\vdash_{S^2} c$ iff $\vdash_{S^2_v \cup R} c$ for any $S^1 \subseteq_F S^2$, we conclude $\Vdash^F_{S^2} A$ if and only if $\Vdash^{F^2}_{S^2_v \cup R} A$, for all $S^1 \subseteq_F S^2$. Since we have previously obtained $\Gamma \Vdash^{F^2}_{S^2_v \cup R} A$ for all $(S^2_v \cup R) \in F^2$, we conclude $\Gamma \Vdash^F_{S^2} A$ for all $S^1 \subseteq_F S^2$. Since $\Vdash^F_{S^1} \Gamma$, by monotonicity we have $\Vdash^F_{S^2} \Gamma$ for all $S^1 \subseteq_F S^2$, hence $\Vdash^F_{S^2} A$ for all $S^1 \subseteq_F S^2$. But S^1 is an arbitrary extension of S in F such that $\Vdash^F_{S^1} \Gamma \cup R^*$, and F is a arbitrary multibase focused on S, so we finally conclude $\Gamma, R^* \Vdash_S A$.

[58] Just as in (PRAWITZ, 2006), we can simply pick an arbitrary application of the rule and switch to any application above it (if there are any). Since deductions are always finite, by repeating this procedure we will eventually reach an application which does not occur below any other.

As for the case of atomic axioms, notice that $\Vdash^{F}_{S^1} R^*$ for $R^* = a$ implies the existence of a deduction showing $\vdash_{S^1} a$, hence it immediately follows that for all $S^1 \subseteq_F S^2$ we have $\vdash_{S^2} c$ iff $\vdash_{S^2_v \cup \{[/a]\}} c$, and the rest of the proof proceeds as before. □

Lemma 3.3.18 $\Gamma \Vdash^0_m A$ if and only if $\Gamma \Vdash_\emptyset A$.

Proof: If $\Gamma \Vdash^0_m A$ then $\Gamma \Vdash_S A$ for all S, so by putting $S = \emptyset$ we have $\Gamma \Vdash_\emptyset A$. For the converse, assume $\Gamma \Vdash_\emptyset A$. Then $\Gamma \Vdash^F A$ for all multibases F focused on $\emptyset$. Now let F' be a multibase focused on S, for any $S \neq \emptyset$, and let F'' be the focused multibase obtained by putting $\emptyset$ at the beginning of F'. F'' is a multibase focused on $\emptyset$, so $\Gamma \Vdash^{F''} A$ and then $\Gamma \Vdash^{F''}_{S'} A$ for all $S' \in F''$. But the value of $\Gamma \Vdash^{F''}_{S'} A$ for each $S' \in F''$ depends only on the value of $\Gamma \Vdash^{F''}_{S''} A$ for $S' \subseteq_{F''} S''$ per isomorphism, and since $\emptyset$ is not a extension of any other set in F'' by omiting it we reobtain F' and conclude $\Gamma \Vdash^{F'}_{S'} A$ for any $S' \in F'$, so $\Gamma \Vdash^{F'} A$. But F' is a arbitrary focused multibase on S, so $\Gamma \Vdash_S A$. But the S is also arbitrary, so $\Gamma \Vdash_S A$ holds for all S, hence $\Gamma \Vdash^0_m A$. □

Theorem 3.3.19 Export holds for production multibases with the standard mapping.

Proof: Assume $\Gamma \Vdash_S A$. By applying Lemma 3.3.17 a sufficient number of times we get $\Gamma, S^* \Vdash_\emptyset A$, so by Lemma 3.3.18 we have $\Gamma, S^* \Vdash A$. Now assume $\Gamma, S^* \Vdash A$. By Lemma 3.3.18 we have $\Gamma, S^* \Vdash_\emptyset A$, so by applying Lemma 3.3.17 a sufficient number of times we get $\Gamma \Vdash_S A$. □

Notice that the proof of Lemma 3.3.17 only works if focused multibases only contain production bases. No features of rules capable of discharge were used in any results on Sections 3.2.3, 3.3.1 or 3.3.2 (except the negative results in Theorems 3.3.8 and 3.3.15), hence they all hold for production multibases. As such, we are in principle allowed to admit only production multibases in the semantics.

This result brings focused validity even closer to Prawitz and Dummett's proposals, since they originally considered only production bases. Rules capable of discharging formulas or other rules are considered undesirable by some inasmuch they bring considerable deductive structure to bases, but they were shown to be essential for the proof of important results for standard S-validity in sources such as (SANDQVIST, 2009)(SANDQVIST, 2015)(PIECHA; SANZ; SCHROEDER-HEISTER, 2015). Our positive results for Export in production multibases and negative results for Export in standard focused multibases show that there might actually be good reasons for considering only production bases. We should keep in mind, however, that failure of Import and Export were proven only for multibases in general; provided further restrictions are imposed on multibases, it is possible to reobtain them even in the presence of rules capable of discharge. This means, of course, that there might be reasons to consider such rules in specific contexts (such as when we desire to obtain semantics for logics stronger than minimal logic).

As mentioned before, it is proven in (PIECHA; SCHROEDER-HEISTER, 2019) that any semantics with all of the desiderata normally associated with S-validity (the results proved in Theorem 3.3.4, Theorem 3.3.5 and its converse, Import and Export) is necessarily incomplete. We have proven that most of the desiderata hold for focused multibases in general. Since Import is only expected to hold in the presence of higher-order bases, all but one of the desiderata (the converse of Theorem 3.3.5, disproved in Theorem 3.3.6) hold in production multibases, so this seems to be the closest we can get to implementing the original idea behind proof-theoretic validity.

There might also be an excellent reason for the failure of the aforementioned desiderata: due to soundness and completeness we have $\Gamma \Vdash_S A$ for all S if and only if $\Gamma \vdash^0_m A$, so our notion of consequence successfully encodes minimal derivability. On the other hand, the implication ($\Vdash_S \Gamma \Rightarrow \Vdash_S A$) seems to induce behaviors characteristic of the notion of *admissibility*. It is easy to use the clauses of Theorem 3.3.5 to prove, for example, ($\Vdash_S A \to (B \vee C) \Rightarrow$ ($\Vdash_S A \to B$) or ($\Vdash_S A \to C$)), something that would be expected of an admissibility notion but not of a derivability one. This would explain why virtually all of the incompleteness results for S-validity show that completeness fails because a consequence that is admissible becomes derivable, since derivability implies admissibility but not the other way around. This won't be proved here, however, so this section ends with a conjecture:

Conjecture 3.3.20 ($\Vdash_S A \Rightarrow \Vdash_S B$) holds for arbitrary S if and only if the rule $[A/B]$ is admissible in minimal logic.

3.3.3 Classical and intuitionistic generalized S-validity

The constraints imposed on multibases in order to obtain semantics for classical and intuitionistic logic have important consequences. In fact, adoption of the consistency constraint leads to the failure of Theorem 3.3.1. On the first step of the proof we construct a base S'' by adding axiomatic rules to S, but in order to do so in an intuitionistic setting we would have to guarantee that S'' is also *consistent*. This cannot be done in general, which leads to a distinct result:

Theorem 3.3.21 $\{a^1, \dots, a^n\} \Vdash^i_S b$ iff either ($\{a^1, \dots, a^n\} \vdash_S b$) or ($\{a^1, \dots, a^n\} \vdash_S \bot$).

Proof:

($\Rightarrow$): We construct a base S'' from the base S by using the procedure shown in the first part of Theorem 3.3.1. If S'' is consistent, the proof proceeds in the same fashion, so we conclude $\{a^1, \dots, a^n\} \vdash_S b$. If S'' is inconsistent, there must be a deduction showing $\vdash_{S''} \bot$. Since it must use one of the new rules (else S would be inconsistent), by applying the same procedure of replacing applications of atomic axioms by assumptions we obtain a deduction showing $\Gamma \vdash_S \bot$ for some $\Gamma \subseteq \{a^1, \dots, a^n\}$, hence $\{a^1, \dots, a^n\} \vdash_S \bot$.

($\Leftarrow$): If $\{a^1, \dots, a^n\} \vdash_S b$, the proof proceeds just like in the second part of Theorem 3.3.1. Now suppose $\{a^1, \dots, a^n\} \vdash_S \bot$, and let F be a intuitionistic multibase focused on S. If there were a $S' \in F$ such that $\Vdash_S \{a^1, \dots, a^n\}$ we could use the deductions showing $\vdash_{S'} a^m$ $(1 \leq m \leq n)$ together with the deduction showing $\{a^1, \dots, a^n\} \vdash_S \bot$ (since it is also a deduction showing $\{a^1, \dots, a^n\} \vdash_{S'} \bot$) to prove $\vdash_{S'} \bot$. Since every intuitionistic base has to be consistent, there can be no such S'. Hence $\{a^1, \dots, a^n\} \Vdash^F_{S'} b$ is satisfied vacuously for any $S' \in F$, whence $\{a^1, \dots, a^n\} \Vdash^i_S b$ by arbitrariness of F. □

It is still the case that validity of formulas in general is reducible to atomic derivability, but the dynamics of this interaction changes. Moreover, no other theorems of Section 3.3 require the construction of bases or multibases in this fashion[59], so they also hold for intuitionistic generalized S-validity. In particular, since $\emptyset$ is a consistent base, Lemma 3.3.18 holds for it. Notice that in Lemma 3.3.17 the $S \cup R$ must be consistent, else $\Gamma \Vdash^i_{S \cup R} A$ would not have been defined, but this does not interfere in the proof of Export because in the statement of the theorem it is already assumed that we consider only S for which the semantic relation is defined. Since Lemmas 3.3.17 and 3.3.18 hold for intuitionistic generalized S-validity, Theorem 3.3.19 also holds for it.

Theorem 3.3.1 would still hold if intuitionistic multibases were defined through explosive bases, since the process of adding rules to such bases preserve explosion. Although the embedding in Definition 3.2.11 no longer works, completeness could be proved using the strategy in (STAFFORD; NASCIMENTO, 2023). This might initially suggest that the use of explosion is preferable, but there are other trade-offs to consider.

Theorem 3.3.22 The following principles hold for any intuitionistic S:

($\bot$) $\nVdash^i_S \bot$;

(EFQ) $\Vdash^i_S (A \wedge \neg A) \to B$.

Proof:

($\bot$) Immediate from Definitions 3.2.34, 3.2.35, 3.2.36 and the clause for atoms in Definition 3.2.3.

[59] When dealing with disjunction in Theorem 3.3.4 we only add vacuous rules to bases, and in other cases we only use bases that have to be consistent due to the assumption that the multibases being considered are intuitionistic. We also do not have to prove consistency for the $S^2 \cup R$ used in Theorem 3.3.19, since it is shown that $\vdash_{S^2 \cup R} c$ holds if and only if $\vdash_{S^2} c$ for every S^2 and the S^2 are consistent by construction, hence when $c = \bot$ we have $\nvdash_{S^2 \cup R} \bot$. Notice that a curious property arises from the proof: if $\{a^1, \dots, a^n\} \vdash_{S^2 \cup I} \bot$ then no deduction showing $\vdash_{S^2 \cup I} c$ can use R, since if it were used we would be able to obtain proofs showing $\vdash_{S^2} a^m$ for every $(1 \leq m \leq 1)$ and use them to show $\vdash_{S^2} \bot$.

(EFQ) There cannot be any intuitionistic multibase F focused on S such that $\Vdash^F_{S'} A \wedge \neg A$ for some $S' \in F$, since by the semantic clauses for conjunction and implication this would lead to $\Vdash^F_{S'} \bot$ and then $\vdash_{S'} \bot$, contradicting the consistency requirement. Hence $(A \wedge \neg A) \Vdash^F_{S'} B$ is vacuously satisfied by all $S' \in F$, as is $\Vdash^F_{S'} (A \wedge \neg A) \to B$, so $\Vdash^F (A \wedge \neg A) \to B$, hence by arbitrariness of the (intuitionistic) F focused on S and arbitrariness of the (intuitionistic) S have $\Vdash^i_S (A \wedge \neg A) \to B$. □

The first principle cannot be recovered if we consider only explosive bases, even though we could recover Sandqvist's semantic clause (cf. clause 2 of 3.2.33). The second principle can be recovered, albeit this would require induction on the subformulas of B.

The strongest argument in favor of using the consistency requirement is that principle ($\bot$) is the main desiderata for the semantics of $\bot$. Principle (EFQ), which expresses the idea of logical explosion, is usually expected to be a *consequence* of the definition of $\bot$, not the definition itself. It has been argued before that the definition of $\bot$ in terms of atomic explosion is one of the weaknesses of most approaches to intuitionistic S-validity, since it leads to a departure from the intuition behind intuitionistic negation (PIECHA; SANZ; SCHROEDER-HEISTER, 2015):

> This fact, that any atom a is validated in some extension of any atomic system, might be considered a fault of validity-based proof-theoretic semantics, since it speaks against the intuitionistic idea of negation $\neg A$ as expressing that A can never be verified.

This idea is recovered if the consistency requirement is used. We have shown this for generalized S-validity here, but it also holds for base-extension semantics (NASCIMENTO; PEREIRA; PIMENTEL, 2023). Theorem 3.3.21 is a natural consequence of these intuitions, so Theorem 3.3.1 should not be expected to hold in general.

Since classical multibases are obtained from intuitionistic multibases without adding new negative requirements, all proofs for intuitionistic generalized S-validity also hold for classical generalized S-validity – except the proof of Export, since now Lemma 3.3.18 does not hold because $\emptyset$ is not a classical base.

But we can also obtain new results:

Theorem 3.3.23 The following principle holds for all classical S:

(EM) $\Vdash^c_S A \vee \neg A$.

Proof: Immediate from Theorem 3.2.47 and Definition 3.2.44. □

Lemma 3.3.24 For any S in any classical focused multibase, $\Gamma \Vdash^F_S A$ if and only if $\nVdash^F_S B$ for some $B \in \Gamma$ or $\Vdash^F_S A$.

Corollary 3.3.25 For any S in any classical focused multibase, $\Vdash^F_S A \to B$ if and only if $\nVdash^F_S A$ or $\Vdash^F_S B$.

Proof: Assume $\Gamma \Vdash^F_S A$. If $\nVdash^F_S B$ for some $B \in \Gamma$, by Lemma 3.2.46 we have $\nVdash^F_{S'} B$ for all $S' \in F$, so $\Gamma \Vdash^F_S A$ holds vacuously. If $\Vdash^F_S B$ for all $B \in \Gamma$ then since $\Gamma \Vdash^F_S A$ we have $\Vdash^F_S A$. For the converse, if $\nVdash^F_S B$ for some $B \in \Gamma$ then by Lemma 3.2.46 we have $\nVdash^F_{S'} B$ for all $S' \in F$, so $\Gamma \Vdash^F_S A$ holds vacuously. If $\Vdash^F_S A$ then by Lemma 3.2.46 we have $\Vdash^F_{S'} A$ for all $S' \in F$, so $\Gamma \Vdash^F_S A$ holds. □

Theorem 3.3.26 $\Gamma \Vdash^c_S A$ *iff* $\Gamma \Vdash^{\langle S \rangle}_S A$.

Proof: From Lemma 3.2.46 it follows that, in every classical multibase F focused on S, for any $S' \in F$ we have $\Vdash^F_{S'} A$ if and only if $\Vdash^F_S A$, so for all such S' we have $\vdash_{S'} a$ if and only if $\vdash_S a$. A simple induction on formulas shows that $\Vdash^F_{S'} A$ for all $S' \in F$ if and only if $\Vdash^{\langle S \rangle}_{S'} A$ for all $S' \in \langle S \rangle$ if and only if $\Vdash^{\langle S \rangle}_S A$, so also $\Gamma \Vdash^F_{S'} A$ for all $S' \in F$ if and only if $\Gamma \Vdash^{\langle S \rangle}_S A$. But then $\Gamma \Vdash^F A$ if and only if $\Vdash^{\langle S \rangle}_S A$, and since F was a arbitrary multibase focused on S we conclude $\Gamma \Vdash^c_S A$ if and only if $\Gamma \Vdash^{\langle S \rangle}_S A$. □

So focused multibases and generalized proof-theoretic validity add nothing to classical proof-theoretic semantics, since we could consider only derivability and validity in S (without considering extensions) from the start.

Although the previously presented proof of Export does not hold for classical multibases, a different proof can be obtained as follows:

Definition 3.3.27 For any classical base S, the set S^c of formulas is defined as follows:

1. $\vdash_S a$ implies $a \in S^c$;
2. $a \vdash_S \bot$ implies $\neg a \in S^c$;
3. No other formula is in S^c

Theorem 3.3.28 $\Gamma, S^c \Vdash^0_c A$ if and only if $\Gamma \Vdash^c_S A$.

Proof: ($\Rightarrow$): Assume $\Gamma, S^c \Vdash^0_c A$. Let F be any multibase focused on S. From the definition of S^c we have that $a \in S^c$ implies $\vdash_S a$, which by Lemma 3.2.46 implies $\Vdash^F_{S'} a$ for any $S' \in F$, and $\neg a \in S^c$ implies $a \vdash \bot$ and thus $\nvdash_S a$, which by Lemma 3.2.46 implies $\nVdash^F_{S'} a$ for any $S' \in F$ and so $\Vdash^F_{S'} \neg a$ vacuously for any $S' \in F$. Hence $\Vdash^F_{S'} S^c$ for any $S' \in F$. From our assumption we have $\Gamma, S^c \Vdash^c_S A$ and so $\Gamma, S^c \Vdash^F A$ for any F focused on S, hence $\Gamma, S^c \Vdash^F_{S'} A$ for any $S' \in F$. Now let S'' be any extension of any S' such that $\Vdash^F_{S''} \Gamma$. Then $\Vdash^F_{S''} \Gamma \cup S^c$ and, since $\Gamma, S^c \Vdash^F_{S'} A$, we also have $\Vdash^F_{S''} A$, so from the arbitrariness of S'' we conclude $\Gamma \Vdash^F_{S'} A$, hence from the arbitrariness of $S' \in F$ and of F we have $\Gamma \Vdash^c_S A$.

($\Leftarrow$): Assume $\Gamma \Vdash^{c}_{S} A$, and let F be an arbitrary classical multibase focused on a arbitrary base. Suppose there is some $S' \in F$ such that $\Vdash^{F}_{S'} \Gamma \cup S^c$. Then $\Vdash^{F}_{S''} \Gamma \cup S^c$ for arbitrary $S'' \in F$ by Lemma 3.2.46. Since $\Vdash^{F}_{S''} S^c$ we have that either $a \in S^c$, and so both $\vdash_S a$ and $\vdash_{S''} a$, or $\neg a \in S^c$, hence it can be easily shown that $\nvdash_S a$ and $\nvdash_{S''} a$. We conclude $\vdash_{S''} a$ if and only if $\vdash_S a$ for all $S'' \in F$. A reasoning similar to that used to prove Theorem 3.3.26 shows that, for any Δ and any B, $\Delta \Vdash^{F}_{S''} B$ for all $S'' \in F$ if and only if $\Delta \Vdash^{\langle S \rangle}_{S} B$. Since from our assumptions it follows that $\Gamma \Vdash^{\langle S \rangle}_{S} A$ we conclude $\Gamma \Vdash^{F}_{S''} A$ for arbitrary $S'' \in F$, and since $\Vdash^{F}_{S''} \Gamma \cup S^c$ and so $\Vdash^{F}_{S''} \Gamma$ we conclude $\Vdash^{F}_{S''} A$ for arbitrary S'', hence $\Gamma, S^c \Vdash^{F}_{S''} A$ for arbitrary S'' and so $\Gamma, S^c \Vdash^{F} A$. If there is no $S' \in F$ such that $\Vdash^{F}_{S'} \Gamma \cup S^c$ we have that $\Gamma, S^c \Vdash^{F}_{S'} A$ is satisfied vacuously for all $S' \in F$, hence $\Gamma, S^c \Vdash^{F} A$. Since this covers all cases we conclude $\Gamma, S^c \Vdash^{F} A$. Since F was a arbitrary classical multibase focused on any arbitrary base, we conclude $\Gamma, S^c \Vdash^{0}_{c} A$. $\square$

3.4 Multibases for predicate logic

The use of domains and interpretation functions makes the structure of first and second-order models much more reliant on model-theoretic tools than their propositional counterparts. We certainly do not want a semantics in which derivability must be supplemented by functions, so atomic bases themselves should ideally be able to provide both domains and interpretations. It is possible to obtain a purely proof-theoretic semantics in this sense, but first we must promote some simplifications of Kripke models.

As noted before, we now use first-order atomic rules, understood as atomic rules with first-order sentences. This may be exemplified as follows:

$$\frac{P}{Q} \qquad \frac{Pab}{Qab} \qquad \frac{Pa \quad Qbcd}{Re} \qquad \frac{Pa \quad \begin{matrix}[Qbc] \\ \vdots \\ Rd\end{matrix}}{Sef}$$

The contributions of this section are aimed at proof-theoretic semantics in general, not just multibase semantics. Most approaches still limit themselves to the propositional case due to a lack of results on how to define purely proof-theoretic predicate semantics. Our definitions can be adapted to other contexts – provided bases or similar structures are available – so they may also be used to extend other semantics to predicate logic.

Naturally, from now on when speaking of multibases we are speaking specifically of first-order multibases.

3.4.1 Simplified domains and literal interpretations

Models are defined over domains, understood simply as sets of objects. After the domains are fixed, an interpretation function shows how we may talk about its objects in a specific language. The intuition behind the use of domains and interpretations is that, to build a model, we fix a set of objects considered of interest, use the constants of the language as names for them (adding new constants in case there are not enough names), and then use predicate constants to name relations between them.

Domains can be given both abstract and concrete readings, as can interpretations of predicates. If we consider a domain $SG = \{$ Apollo, Huitzilopochtli, Ra, ... $\}$, understood as the set of all mythological sun gods, a interpretation function can be defined such that *h* is used to name the god Huitzilopochtli, A is used to name the predicate "is a Aztec god", and h is included in the set of objects satisfying A, making Ah ("Huitzilopochtli is a Aztec god") true. The elements of domains are defined as abstract objects, so we do not need to specify their meanings in this manner when investigating general properties of a model. Once we start dealing with models intended to have concrete meanings, however, it becomes necessary to also provide meaning to the elements.

If we are interested only in concrete models, the description just provided contains a redundancy. Since sets are abstract representations of collections, objects of SG are clearly *names* or *representations* of sun gods, not the sun gods themselves. The expression "understood as the set of all mythological sun gods" is not entirely accurate, even though it successfully conveys the model's intended meaning. This makes it so that, when assigning constants to elements of a domain, we are essentially *naming other names*.

The redundancy can be eliminated if we *directly add the names of the set to the language* instead of using other names for them. However, this would lead to awkward and heterogeneous notation, so we should ideally simplify the names somehow. Since domains are defined as arbitrary sets of objects, we can partly maintain the model-theoretic tradition of regarding constants as names by considering only *domains that are sets of constants*. This way, we start with domains only containing constants, add them to the language and then define interpretations in which each constant is named by itself.

From these ideas we obtain the following definitions:

Definition 3.4.1 A domain is *simplified* if all its members are individual constants.

Definition 3.4.2 A domain assignment function α (cf. Definition 2.3.7) is *simplified* if it only assigns simplified domains.

Definition 3.4.3 Let $\mathbb{L}$ be a language, W a set of objects k and α a simplified domain assignment function. An interpretation function β for them (cf. Definition 2.3.8) is *literal* if $a \in \alpha(W)$ and $a \in \mathbb{L}$ implies $\beta(a) = a$.

Definition 3.4.4 A *pure first-order language* $\mathbb{P}$ is a first-order language (cf. Definition 2.1.2) with no individual constants.

Literal functions use constants to name themselves whenever they are both an object in the domain of W and a constant of the language $\mathbb{L}$. But if a constant of the domain does not occur in the language, it must be named by some other constant, and if a constant of the language does not occur on the domain, it must name another constant. In order to make it so that every constant names itself and no other constant names it[60], when defining a model with objects W we start with a pure language and use the extended language $\mathbb{P} \cup \alpha(W)$ instead of the arbitrary extensions of Definition 2.3.9.

Definition 3.4.5 A first-order minimal model K with objects W and language $\mathbb{P} \cup \alpha(W)$ (cf. Definition 2.3.10) is *simplified* if α is a simplified domain assignment function and β is a literal interpretation function.

We now prove that nothing is lost if we consider only simplified models:

Theorem 3.4.6 Let K be a first-order minimal model for a extended language $\mathbb{L}(\alpha(W))$, and let W be its set of objects k. There is a simplified first-order minimal model K' for the language $\mathbb{L}(\alpha(W))$ with a set W' of objects k' such that $\Vdash_k^K A$ if and only if $\Vdash_{k'}^{K'} A$.

Proof: We start by picking any W' with the same number of objects as W, and then construct the simplified α' and literal β' as follows:

1. $a \in \alpha'(k')$ if and only if $\beta(a) \in \alpha(k)$.
2. $\langle a^1, ..., a^n \rangle \in \beta'(P_n, k')$ if and only if $\langle \beta(a^1), ..., \beta(a^n) \rangle \in \beta(P_n, k)$.

In other words, the domain of each $k' \in W'$ is the set of all constants assigned to some object of the domain of k. Since all constants of $\mathbb{L}(\alpha(W))$ must have been assigned to some element of $\alpha(W)$, we have that $\alpha'(W')$ is just the set of all individual constants in $\mathbb{L}(\alpha(W))$, so this is indeed a language $\mathbb{P} \cup \alpha'(W')$. The interpretation that β' gives to each individual constant is determined by the fact that it is literal, and it also includes in the n-ary relation $\beta'(P_n, k')$ assigned to each pair (P_n, k') a n-tuple $\langle a^1, \ ... \ a^n \rangle$ of constants whenever the objects originally assigned to each constant were also in a similar tuple included in the relation $\beta(P_n, k)$. The number of tuple increases if more than one constant was originally used to name an object (as

[60] This might not be ideal in some contexts. If we wish to define $a = b$ as holding whenever $\beta(a) = \beta(b)$, fullfilment of this requirement implies that every constant is equal only to itself. This is not problematic if we define equality in terms of transitive, reflexive and symmetric relations, but definitions in terms of interpretations require at least that we let constants not on the domain to occur in the language.

does the domain), but the construction process guarantees that all tuples of constants behave exactly as the tuples of objects they were originally assigned to.

Let v be the valuation function of K and v' be the function of K'. We first stipulate that, for all 0-ary predicates, $v(P_0) = T$ iff $v'(P_0) = T$.

Now assume $\Vdash_k^K P_n(a^1, ..., a^n)$. Then $v(P_n(a^1, ..., a^n), k) = T$, so $\langle \beta(a^1), ..., \beta(a^n) \rangle \in \beta(P_n, k))$. By our definition of β' we have $\langle a^1, ..., a^n \rangle \in \beta'(P_n, k')$. Since from the fact that β' is literal we have $\beta'(a) = a$ for all a, it also holds that $\langle \beta'(a^1), ..., \beta'(a^n) \rangle \in \beta'(P_n, k')$, thus $v'(P_n(a^1, ..., a^n), k') = T$, hence $\Vdash_{k'}^{K'} P_n(a^1, ..., a^n)$. The converse can be proved in a similar fashion.

The induction for formulas without quantifiers is straightforward. For quantified formula, it proceeds as follows:

1. Assume $\Vdash_k^K \forall x A$. Then $\Vdash_j^K A[x/a]$ for all a such that $\beta(a) \in \alpha(j)$ in all $k \leq j$. Induction hypothesis: $\Vdash_{j'}^{K'} A[x/a]$ for all a such that $\beta(a) \in \alpha(j)$ in all $k' \leq j'$. But by definition we have $\beta'(a) \in \alpha'(j')$ if and only if $a \in \alpha'(j')$ if and only if $\beta(a) \in \alpha(j)$, so it holds that $\Vdash_{j'}^{K'} A[x/a]$ for all a such that $\beta'(a) \in \alpha'(j')$ in all $k' \leq j'$, hence $\Vdash_{k'}^{K'} \forall x A$. The converse can be proved in a similar fashion.

2. Assume $\Vdash_k^K \exists x A$. Then $\Vdash_k^K A[x/a]$ for some a such that $\beta(a) \in \alpha(k)$. Induction hypothesis: $\Vdash_{k'}^{K'} A[x/a]$ for some a such that $\beta(a) \in \alpha(k)$. We can show $\beta'(k') \in \alpha'(k')$ if and only if $\beta(k) \in \alpha(k)$ once again, so $\Vdash_{k'}^{K'} A[x/a]$ for some a such that $\beta'(a) \in \alpha'(k')$, hence $\Vdash_{k'}^{K'} \exists x A$. The converse can be proved in a similar fashion. □

Corollary 3.4.7 There is a model K such that $\Gamma \vDash^K A$ in the language $\mathbb{L}$ if and only if there is a simplified model K' with $\Gamma \vDash^{K'} A$ in the language $\mathbb{L}$.

Corollary 3.4.8 First-order minimal logic is sound and complete with respect to first-order simplified Kripke models.

Proof: The left-to-right direction of the first corollary follows from Theorem 3.4.1, and the right-to-left direction follows from the fact that a simplified model is still a model. The soundness and completeness results follow immediately from the equivalence. □

We have thus proved that constants included in domains may be later used to name themselves. Notice that we do not extract the constants from any fixed language, so this is not the same as requiring every constant of the domain to already have a name[61]. In fact, this constraint cannot be imposed: by requiring all objects to have preexisting names we obtain incompleteness results (ARRUDA; MARTINS; PEREIRA, 2012).

[61] I am thankful to Prof. Luiz Carlos Pereira for pointing out that this relation was in need of clarification.

To see why, consider the invalid consequence $(\{Pa^1, Pa^2, ...\} \cup \exists x \neg Px) \nvDash^1_m \bot$. If we introduce one object e^n in all domains for every constant a^n with $1 \leq n$ and put $\beta(a^n) = e^n$, put $e^n \in \beta(P, k)$ for all $k \in K$ and all such n, put $\nu(\bot, k) = \emptyset$ for all k, introduce a new object e in the domain of all k such that $e \notin \beta(P, k)$ and add a new constant b to the language to name it, we get a model making all formulas in $(\{Pa^1, Pa^2, ...\} \cup \exists x \neg Px)$ true, but not $\bot$. Now assume there is a language $\mathbb{L}$ with constants a^n and such that for, any element e of any domain, there is some $a^m \in \mathbb{L}$ $(1 \leq m)$ such that $\beta(a^m) = e$. Pick any model K. Since for every $e \in \alpha(K)$ we have $\beta(a^m) = e$ for some m, for $\{Pa^1, Pa^2, ...\}$ to be true we must have $e \in \beta(P, k)$ for all k. But now we cannot add an object e' to the domain of any k such that $e' \notin (P, k)$ for some k, since we would have some a^m such that $\beta(a^m) = e'$, hence $e' \in (P, k)$. We conclude that no model making $(\{Pa^1, Pa^2, ...\})$ true in this language can also make $\exists x \neg Px$ true, so the consequence $(\{Pa^1, Pa^2, ...\} \cup \exists x \neg Px) \vDash^1_m \bot$ holds vacuously. Failure of completeness ensues, as do failure of properties such as compactness. We conclude that it is not possible to consider only languages in which every conceivable object already has a name.

This is different from our definitions because we have not assumed that the constants were obtained from some language, so it is always possible to add to the domain some new constant that is not in the language we are considering. In other words, our supply of constants is limited neither by our choice of domain nor by our choice of language, and the problem only appears under the assumption that the language limits the supply of new (unnamed) objects. If we have a language $\mathbb{L}$ containing all constants a^n for $1 \leq n$ and consider a series of formulas $\{Pa^1, Pa^2, ...\}$, we can always add a new constant b to the domain of all k and put $b \notin \beta(P, k)$ to obtain a model making $(\{Pa^1, Pa^2, ...\} \cup \exists x \neg Px)$ true in the language $\mathbb{L} \cup b$. Constants receive the same treatment usually dispensed to elements of sets, so the problem is avoided.

In light of this, the important part is that the following property is satisfied:

Definition 3.4.9 (Fresh constants requirement) For every language $\mathbb{L}$, there is some constant a such that $a \notin \mathbb{L}$ and $\mathbb{L} \cup a$ is also a language.

So after adding every constant of the domains to the pure language we would still have some constant not in the new language, which could be added to the original domain of the model in order to extend it and then also added to the language. This new constant cannot have a preexisting (literal) name in the language precisely because it was not in the language. Paradoxes of this kind are thus avoided.

3.4.2 Natural domains

In propositional bases, the semantic properties of propositions are entirely determined by how they may be used together with rules to prove other propositions. The meaning of a proposition is fixed by its inferential content in some relevant sense. This is also expected of first-order bases, so the meaning of a first-order sentence should also be fixed by its inferential content in some relevant sense. The same reasoning applies to domains and interpretations, since they are also determinants of the meaning of first-order sentences. We will later show that interpretation functions can be replaced by first-order atomic derivability, but for now we focus on first-order domains.

The domain of discourse is defined in order to specify to which objects the semantics is being applied to. Due to the shape of their premises and conclusions, first-order rules establish derivability relations between first-order sentences even before the individual constants occurring in them are interpreted. As seen before, a constant can be interpreted by itself if we adopt literal interpretations, and in this case each sentence of a derivation would be affirming something of the constants occurring in it. But, since the sentences are affirming things of the constants themselves, we can already determine a domain of discourse: the set of all objects of which something can be affirmed when an inference licensed by the base is made – that is, *the set of all individual constants appearing in premises, conclusions, or dischargeable formulas of rules of the base*.

This idea contains a single *caveat*: the logics we are working with only admit non-empty domains, but the domain of discourse of a base may be empty (e.g. nothing is affirmed of anything in the empty base, so its domain is empty)[62]. This can be solved either by fixing a "special" constant, which must be in the domain of discourse even if it is not used by any rules[63], or requiring all bases to contain at least one rule making use of a individual constant. Both options are equally functional, but we adopt the latter due to the model-theoretic character of the former.

The idea is formalized as follows:

Definition 3.4.10 (Existential import) A multibase satisfies the property of *existential import* if every base in it has some rule in which at least one individual constant occurs in a premise, conclusion, or dischargeable formula.

From now on we assume that all multibases satisfy existential import.

Definition 3.4.11 For any base S, its natural domain $N(S)$ is the set of all individual constants occurring in premises, conclusions, or dischargeable formulas of rules in S.

Definition 3.4.12 For any multibase M, its natural domain $N(M)$ is $\bigcup_{S \in M} N(S)$.

[62] This means that a more general framework is obtained if we consider *inclusive* versions of each logic as starting points, since empty domains would then become acceptable (NOLT, 2021).

[63] Focused multibases for S would only require one special constant, since the domain of discourse of S must be a subset of every other domain of discourse and so there must be at least one constant shared by all domains. This is not a feature of multibases in general, so it might be desirable to consider multiple selected constants when dealing with standard validity.

A desirable consequence follows:

Theorem 3.4.13 $S \subseteq S'$ implies $N(S) \subseteq N(S')$.

Proof: Let S and S' be any bases such that $S \subseteq S'$. Then every rule of S is also a rule of S', so every constant occurring in some rule of S also occurs in some rule of S'. But this makes it so that every constant extracted from the rules of S is also extracted from the inferences of S', so $N(S) \subseteq N(S')$. □

The natural domain of a base is just the set of all individual constants its rules use, and that of a multibase is just the union of the natural domains of its bases. As a consequence, the content of each domain is entirely determined by the rules of the corresponding base, so the definition is purely proof-theoretic. Preservation of domains is then a direct consequence of preservation of rules by the extension relation.

3.4.3 First-order logic

Natural domains are the proof-theoretic counterparts of model-theoretic domain assignment functions, but we must still provide a proof-theoretic version of interpretation functions. Literal interpretations already eliminate the need for explicit interpretations of individual constants, but models still need the function to interpret predicate constants. However, the sole purpose of predicate interpretations is to establish for which tuples of objects each predicate holds, and this can be done by *atomic derivability* itself.

We are now ready to define validity for first-order multibase semantics:

Definition 3.4.14 The (first-order) relations of base validity ($\Vdash_S^M$), multibase validity ($\Vdash^M$), standard validity ($\Vdash$), generalized S-validity ($\Vdash_S$) and focused validity ($\Vdash_m^1$) are defined as follows, for $S \in M$ and in the language $\mathbb{P} \cup N(M)$

1. $\Vdash_S^M Pa^1 ... a^n \iff \vdash_S Pa^1 ... a^n$;

2. Clauses 2 through 8 are as in Definition 3.2.3;

9. $\Vdash_S^M \forall x(A) \iff \forall S'(S \subseteq S') : \Vdash_{S'}^M A[x/a]$, for all $a \in N(S')$;

10. $\Vdash_S^M \exists x(A) \iff \Vdash_S^M A[x/a]$, for some $a \in N(S)$;

11. $\Gamma \Vdash_S A$ iff $\Gamma \Vdash^F A$ for all first-order multibases F focused on S;

12. $\Gamma \Vdash_m^1 A$ iff $\Gamma \Vdash_S A$ for all S.

The interpretation of individual constants is implicitly taken to be literal, and the n-tuples for which the predicates hold is given by derivability in the first-order base instead of by interpretations. The simplified, natural domains are also extracted directly from the structure of bases. There are no reminiscences of model theory in this semantics, so we conclude that this is indeed an entirely proof-theoretic first-order semantics.

We now prove completeness for this semantics by adapting Definitions 3.2.11 and 3.2.17 to the first-order case.

Definition 3.4.15 For any first-order focused multibase M, its corresponding simplified first-order minimal Kripke model K^M is defined as follows:

1. The set W of K^M is the set of all bases occurring in M;
2. $S \leq S'$ if and only if $S \subseteq_M S'$.
3. For any 0-ary predicate P, $v(P, S) = T$ if and only $\vdash_S P$;
4. For any $(n+1)$-ary predicate P_n, $\langle a^1, \ldots, a^n \rangle \in \beta(P, S)$ if and only if $\vdash_S Pa^1 \ldots a^n$;
5. $\alpha(S) = N(S)$;
6. β is literal.

From $\alpha(S) = N(S)$ it follows that the domains are simplified, and from the definitions of domains for models and multibases we have $\alpha(M) = N(M)$. Since the multibase is defined for the language $\mathbb{P} \cup N(M)$ the language is also $\mathbb{P} \cup \alpha(M)$, hence a literal β is available. From our construction and Theorem 3.4.13 if follows that $S \leq S'$ implies $\alpha(S) \subseteq \alpha(S')$. From preservation of rules by extensions we also have that $\vdash_S Pa^1 \ldots a^n$ and $S \subseteq S'$ implies $\vdash_{S'} Pa^1 \ldots a^n$, so due to our construction $S \leq S'$ and $\langle a^1, \ldots a^n \rangle \in S$ implies $\langle a^1, \ldots a^n \rangle \in S'$, hence $S \leq S'$ implies $\beta(P, S) \subseteq \beta(P, S')$. This means that, aside from the specific conditions of simplified models, all conditions of Definition 2.3.10 are satisfied, so K^M is indeed a first-order simplified minimal Kripke model.

Theorem 3.4.16 $\Vdash^M_S A \iff \vDash^{K^M}_S A$.

Proof:

1. Atomic case: for 0-ary predicates the result is immediate. For $(n+1)$-ary predicates, in any $S \in M$, $\Vdash^M_S Pa^1 \ldots a^n$ implies $\vdash_S Pa^1 \ldots a^n$ implies $\langle a^1, \ldots, a^n \rangle \in \beta(P, S)$ implies $v(Pa^1 \ldots a^n, S) = T$, which implies $\vDash^{K^M}_S Pa^1 \ldots a^n$. For the converse, $\vDash^{K^M}_S Pa^1 \ldots a^n$ implies $v(Pa^1 \ldots a^n, S) = T$ implies $\langle a^1, \ldots, a^n \rangle \in \beta(P, S)$, but due to our construction procedure this must be because $\vdash_S Pa^1 \ldots a^n$, hence $\Vdash^M_S Pa^1 \ldots a^n$.
2. $(\wedge)$, $(\vee)$ and $(\to)$: identical to the proof in Theorem 3.2.12.

3. ($\forall$): Let $\Vdash^M_S \forall x A$. Then for all $S \subseteq_M S'$ we have $\Vdash^M_{S'} A[x/a]$ for all $a \in N(S')$. Induction hypothesis: for all $S \subseteq_M S'$ we have $\vDash^{KM}_{S'} A[x/a]$ for all $a \in N(S')$. But $S \subseteq_M S'$ if and only if $S \leq S'$ and $N(S') = \alpha(S')$, hence $S \leq S'$ implies $\vDash^{KM}_{S'} A[x/a]$ for all $a \in \alpha(S')$, hence $\vDash^{KM}_S \forall x A$. For the converse, let $\vDash^{KM}_S \forall x A$. For all $S \leq S'$ we have $\vDash^{KM}_{S'} A[x/a]$ for all $a \in \alpha(S')$. Induction hypothesis: for all $S \leq S'$ we have $\Vdash^M_{S'} A[x/a]$ for all $a \in \alpha(S')$. Since $S \leq S'$ if and only if $S \subseteq_M S'$ and $\alpha(S') = N(S')$, $S \subseteq_M S'$ implies $\Vdash^M_{S'} A[x/a]$ for all $a \in N(S')$, hence $\Vdash^M_S \forall x A$.

4. ($\exists$): Let $\Vdash^M_S \exists x A$. Then $\Vdash^M_S A[x/a]$ for some $a \in N(S)$. Induction hypothesis: $\vDash^{KM}_S A[x/a]$ for some $a \in N(S)$. But $N(S) = \alpha(S)$, so $\vDash^{KM}_S A[x/a]$ for some $a \in \alpha(S)$, hence $\vDash^{KM}_S \exists x A$. For the converse, let $\vDash^{KM}_S \exists x A$. Then $\vDash^{KM}_S A[x/a]$ for some $a \in \alpha(S)$. Induction hypothesis: $\Vdash^M_S A[x/a]$ for some $a \in \alpha(S)$. But $N(S) = \alpha(S)$, so again $\vDash^{KM}_S A[x/a]$ for some $a \in N(S)$, hence $\Vdash^M_S \exists x A$. □

Corollary 3.4.17 $\Vdash^M A \iff \vDash^{KM} A$.

Corollary 3.4.18 $\Gamma \vDash^1_m A$ implies $\Gamma \Vdash A$.

Proof: Identical to that of Theorem 3.2.14. □

Definition 3.4.19 A *structural vacuous rule* is a atomic rule concluding an 0-ary predicate from itself (e.g. $[P/P]$, $[Q/Q]$)

Definition 3.4.20 A *discursive vacuous rule* is a atomic rule concluding an sentence with a 1-ary predicate from itself (e.g. $[Pa/Pa]$, $[Qa/Qa]$)

Due to our definition of natural domains, we must use 0-ary predicates in order to avoid including unwarranted constants on domains when inducing a specific order through structural vacuous rules. Discursive vacuous rules are used to induce precisely the domain we want in each base, since they add nothing in terms of derivability but contribute to the natural domain of the base.

Definition 3.4.21 For any first-order simplified minimal model K, a corresponding multibase M^K for it defined as follows:

1. The sequence of M contains one system S^k for each $k \in W$, and no other;

2. If $v(P, k) = T$ for 0-ary P, we add to S^k the atomic axiom $[/P]$;

3. If $\langle a^1, ..., a^n \rangle \in \beta(P, k)$, we add to S^k the atomic axiom $[/Pa^1 \ ... \ a^n]$;

4. If $k \leq k'$, we add the structural vacuous rule $[P^k/P^k]$ to $S^{k'}$.

5. Let Q be a fixed 1-ary predicate. If $a \in \alpha(k)$ then $[Qa/Qa] \in S^k$;

6. Each S^k contains no rules other than those added by the above procedures.

From the fact that the domain of each k is non-empty it follows that the multibase satisfies existential import, since for every S^k there will be a rule $[Qa/Qa] \in S^k$.

Lemma 3.4.22 $k \leq k'$ if and only if $S^k \subseteq_{M^k} S^{k'}$.

Proof:

($\Rightarrow$): Assume $k \leq k'$. Then S^k and $S^{k'}$ are constructed by the addition of axioms, structural vacuous rules and discursive vacuous rules for k and k', respectively. Due to the heredity condition, for 0-ary P we have that if $v(P, k) = T$ then $v(P, k') = T$, hence all 0-ary atomic axioms in S^k are also in $S^{k'}$. We also have that $\langle a^1, \ ... \ a^n \rangle \in \beta(P, k)$ and $k \leq k'$ implies $\langle a^1, \ ... \ a^n \rangle \in \beta(P, k')$, so all n-ary axiom rules of S^k are also in $S^{k'}$.

As for the discursive vacuous rules, notice that $k \leq k'$ implies $\alpha(k) \subseteq \alpha(k')$ and so $a \in \alpha(k)$ implies $a \in \alpha(k')$, hence $[Qa/Qa] \in S^k$ implies $[Qa/Qa] \in S^{k'}$.

Now assume that there is some structural vacuous rule $[P^{k''}/P^{k''}]$ in k. Just as in the propositional case, we have $k'' \leq k$ and, due to transitivity of $\leq$ and our assumption, $k'' \leq k'$ holds, hence the vacuous rule is also in $S^{k'}$. Since all atomic axioms and vacuous rules of S^k are in $S^{k'}$, we conclude $S^k \subseteq_{M^K} S^{k'}$.

($\Leftarrow$): Assume $S^k \subseteq_M S^{k'}$. Since every structural vacuous rule $[P^{k''}/P^{k''}]$ in S^k is also in $S^{k'}$, by the structure of the procedure for adding vacuous rules we conclude that $k'' \leq k$ implies $k'' \leq k'$, for every k''. By reflexivity of $\leq$ we have $k \leq k$, thus $k \leq k'$. □

Lemma 3.4.23 $\alpha(k) = N(S^k)$.

Proof: For every $a \in \alpha(k)$ we have $[Qa/Qa] \in S^k$, so by the definition of natural domains $a \in \alpha(k)$ implies $a \in N(S^k)$. For every $a \in N(S^k)$, due to the structure of S^k, either a occurs in a atomic axiom or in a discursive vacuous rule. If $[Qa/Qa] \in S^k$ then the rule was added because $a \in \alpha(k)$. If $[/Pa^1, \ ... \ a^n] \in S^k$ then $\langle a^1, \ ... \ a^n \rangle \in \beta(P, k)$, but since all elements of $\beta(P, k)$ are sets of tuples of elements in $\alpha(k)$ we have $a^m \in \alpha(k)$ for every $(1 \leq m \leq n)$. But then every individual constant occurring in a axiomatic rule or vacuous discursive rule of S^k is also in $\alpha(k)$, so $a \in N(S^k)$ implies $a \in \alpha(k)$, and since all elements of $\alpha(k)$ are in $N(S^k)$ and vice-versa we conclude $\alpha(k) = N(S^k)$. □

Lemma 3.4.24 $\vDash_k^K A \iff \Vdash_{S^k}^{M^K} A$.

Proof:

1. 0-ary predicates: $v(P, k) = T$ implies $\vdash_{S^k} P$ by construction, so $\vDash_k^K P$ implies $\Vdash_{S^k}^{M^K} P$. For the other direction, notice that all 0-ary axiomatic rules in S^k are added by the procedure and that $\vdash_{S^k} P$ holds only if $[/P]$ was added to S^k, thus $\vdash_{S^k} P$ implies $v(P, k) = T$, so $\Vdash_{S^k}^{M^K} P$ implies $\vDash_k^K P$.

2. $(n+1)$-ary predicates: $v(Pa^1 \ldots a^n, k) = T$ implies $\langle a^1, \ldots, a^n \rangle \in \beta(P, k)$, which by construction implies $[/Pa^1 \ldots a^n] \in S^k$, hence $\vdash_{S^k} Pa^1 \ldots a^n$ and $\Vdash^{M^K}_{S^k} Pa^1 \ldots a^n$, so $\vDash^K_k Pa^1 \ldots a^n$ implies $\Vdash^{M^K}_{S^k} Pa^1 \ldots a^n$. For the other direction, notice that all $(n+1)$-ary atomic axioms in S^k are added by the procedure and that $\vdash_{S^k} Pa^1 \ldots a^n$ holds only if $[/Pa^1 \ldots a^n]$ was added to S^k, so $\vdash_{S^k} Pa^1 \ldots a^n$ implies $\langle a^1, \ldots, a^n \rangle \in \beta(P, k)$ implies $v(Pa^1, \ldots, a^n, k) = T$ implies $\vDash^K_k Pa^1 \ldots a^n$, hence $\Vdash^{M^K}_{S^k} Pa^1 \ldots a^n$ implies $\vDash^K_k Pa^1 \ldots a^n$

3. $(\wedge)$, $(\vee)$ and $(\rightarrow)$: just like in the propositional case, the proof proceeds as in Theorem 3.2.12, the only differences being that we substitute $\vDash^{KM}_S$ by $\vDash^K_k$, $\Vdash^M_S$ by $\Vdash^{M^K}_{S^k}$ and that, in the proof for $A \rightarrow B$, the equivalence $k \leq k'$ iff $S^k \subseteq_{M^K} S^{k'}$ holds due to Lemma 3.4.22 instead of by definition.

4. $(\forall)$ and $(\exists)$: identical to the proof of Theorem 3.4.16, but instead of holding by definition $\alpha(k) = N(S^k)$ holds due to Lemma 3.4.23, as does $k \leq k'$ iff $S^k \subseteq_{M^K} S^{k'}$ due to Lemma 3.4.22. □

Corollary 3.4.25 $\Gamma \vDash^K A \iff \Gamma \Vdash^{M^K} A$.

Lemma 3.4.26 $\Gamma \Vdash A$ implies $\Gamma \vDash^1_m A$.

Proof: Identical to that of Lemma 3.2.21. □

Theorem 3.4.27 (Soundness and completeness) $\Gamma \vDash^1_m A \iff \Gamma \Vdash A$.

Theorem 3.4.28 (Focused soundness and completeness) $\Gamma \vDash^1_m A \iff \Gamma \Vdash^1_m A$.

Proof: Identical to the proofs of Theorems 3.2.25 and 3.2.30. □

Just like before, the result also holds for finite multibases and focused multibases.

3.4.4 Second-order logic

The extension of the results for first-order logic to second-order logic is mostly straightforward. Some features of constructions must be adapted, but almost every step of every proof is a straightforward adaptation of the corresponding first-order result.

We start by expanding the definitions of simplified models to second-order logic.

Definition 3.4.29 A n-ary domain is *simplified* if it is a set of n-ary predicate constants.

Definition 3.4.30 A generalized domain assignment function is *simplified* if the domains $\alpha(k)$ it assigns to each k are simplified domains, the n-ary domains $\alpha^n(k)$ it assigns to each k are simplified n-ary domains and the condition that $\bot \in \alpha^0(k)$ for all k is satisfied.

Definition 3.4.31 Let $\mathbb{L}$ be a language, W a set of objects k and α a simplified generalized domain assignment function. A generalized interpretation function for them (cf. Definition 2.4.4) is *literal* if $a \in \alpha(W)$ and $a \in \mathbb{L}$ implies $\beta(a) = a$ and $P_n \in \alpha^n(W)$ and $P_n \in \mathbb{L}$ implies $\beta(P_n) = P_n$.

Definition 3.4.32 A *pure second-order language* $\mathbb{P}$ is a second-order language (cf. Definition 2.1.3) with no individual constants and no n-ary constants aside from $\bot$.

Definition 3.4.33 A second-order minimal model K for the language $\mathbb{P} \cup \alpha^*(W)$ is simplified if α is a simplified generalized domain assignment function and β is a literal generalized domain assignment function.

Theorem 3.4.34 Let K be a second-order minimal model for a extended language $\mathbb{L}(\alpha^*(W))$ (cf. Definition 2.4.5), and let W be its set of objects k. There is a simplified second-order minimal model K' for the language $\mathbb{L}(\alpha^*(W))$ with a set of objects k' such that $\Vdash_k^K A$ if and only if $\Vdash_k^K A$.

Proof: The proof is very similar to that of Theorem 3.4.6. The construction of K' is as follows:

1. $a \in \alpha'(k')$ if and only if $\beta(a) \in \alpha(k)$;
2. $P_n \in \alpha^n(k')$ if and only if $\beta(P_n) \in \alpha^n(k)$;
3. If $P_n \in \alpha^n(k)$ then $\langle a^1 \, ... \, a^n \rangle \in \beta'(P_n, k')$ iff $\langle \beta(a^1), \, ... \, \beta(a^n) \rangle \in \beta(\beta(P_n), k)$.

Let *b* be the valuation function of K and *v'* the function of K'. If $\beta(P_0) \notin \alpha^0(k)$ then $v(P_0, k) = \emptyset$, but then by construction $P_0 \notin \alpha^0(k')$ and since $\beta(P_0) = P_0$ also $v(P_0, k) = \emptyset$. If $\beta(P_0) \in \alpha^0(k)$ then also $P_0 \in \alpha^0(k)$, and since we are now free to choose the value of P_0 we stipulate that $v(P_0, k) = T$ if and only if $v(P_0, k) = T$.

Now assume $\Vdash_k^K P_n(a^1, ..., a^n)$. Then $v(P_n(a^1, ..., a^n), k) = T$, so $\langle \beta(a^1), ..., \beta(a^n) \rangle \in \beta(\beta(P_n), k))$, so also $(\beta(P_n), k) \in \alpha^n(k)$ (cf. the clause for *v* in Definition 2.4.7). By our definition of β' and $\alpha^n(k')$ we have $P_n \in \alpha^n(k')$, and also $\langle a^1, ..., a^n \rangle \in \beta'(P_n, k')$. Since from the fact that β' is literal we have $\beta'(a) = a$ for all a and also $\beta'(P_n) = P_n$, it also holds that $\langle \beta'(a^1), ..., \beta'(a^n) \rangle \in \beta'(\beta'(P_n), k')$, thus $v'(P_n(a^1, ..., a^n), k') = T$, hence $\Vdash_{k'}^{K'} P_n(a^1, ..., a^n)$. The converse can be proved in a similar fashion.

All steps of the induction proceeds exactly as in Theorem 3.4.6, second-order quantifiers receiving the same treatment as first-order quantifiers. □

Definitions of natural domains must also be adapted:

Definition 3.4.35 (Generalized existential import) A multibase satisfies the property of *generalized existential import* if every base in it has some rule in which at least one individual constant occurs in a premise, conclusion or dischargeable formula, some rule in which at least one $(n+1)$-ary predicate constant occurs in a premise, conclusion of dischargeable formula, and some rule in which $\bot$ occurs in a premise, conclusion of dischargeable formula.

Adding new 0-ary constants is entirely optional because we must already guarantee that $\bot$ is in the 0-ary domain of every base, so the 0-ary domains will never be empty.

Definition 3.4.36 Given any first-order base S, its natural domain $N(S)$ is the set of all individual constants occurring in premises, conclusions, or dischargeable formulas of rules in S, and its n-ary natural domains $N^n(S)$ are the sets of all n-ary predicate constants occurring in premises, conclusions, or dischargeable formulas of rules in S.

Definition 3.4.37 The natural domain $N(M)$ of a multibase M is $\bigcup\limits_{S\in M} N(S)$, and its natural n-ary domains are $\bigcup\limits_{S\in M} N^n(S)$. Its total domain $T(S)$ is the union of its natural domain and all its natural n-ary domains.

As in the first-order case, we have:

Theorem 3.4.38 $S \subseteq S'$ implies $N(S) \subseteq N(S')$ and $N^n(S) \subseteq N^n(S')$, for every n.

Proof: Straightforward adaptation of the proof of Theorem 3.4.13. □

Second-order validity for second-order multibases (understood as first-order multibases satisfying generalized existential import) may be defined as follows:

Definition 3.4.39 The (second-order) relations of base validity ($\Vdash^M_S$), multibase validity ($\Vdash^M$), standard validity ($\Vdash$), generalized S-validity ($\Vdash_S$) and focused validity ($\Vdash^2_m$) are defined as follows, for $S \in M$ and in the language $\mathbb{P} \cup T(M)$:

1. Clauses 1 through 10 are as in Definition 3.4.14;

11. $\Vdash^M_S \forall X_n(A) \iff \forall S'(S \subseteq S') : \Vdash^M_{S'} A[X_n/P_n]$, for all $P_n \in N^n(S')$;

12. $\Vdash^M_S \exists X_n(A) \iff \Vdash^M_S A[X_n/P_n]$, for some $P_n \in N^n(S)$;

13. $\Gamma \Vdash_S A$ iff $\Gamma \Vdash^F A$ for all second-order multibases F focused on S;

14. $\Gamma \Vdash^2_m A$ iff $\Gamma \Vdash_S A$ for all S.

The proof of completeness can also be adapted as follows:

Definition 3.4.40 For any second-order focused multibase M, its corresponding simplified second-order minimal Kripke model K^M is defined as follows:

1. Clauses 1 through 6 are defined as in 3.4.15;

7. $\alpha^n(S) = N^n(A)$, for all n

This can be shown to be a second-order model by the same reasoning used to show the result for first-order multibases.

Theorem 3.4.41 $\Vdash_S^M A \iff \vDash_S^{KM} A$.

Proof:

1. Atomic case, $(\wedge)$, $(\vee)$ and $(\rightarrow)$, $(\forall x)$ and $(\exists x)$ are identical to the proof for Theorem 3.4.16;

2. $(\forall X_n)$ and $(\exists X_n)$: straightforward adaptation of the proof for $(\forall x)$ and $(\exists x)$ in Theorem 3.4.16 □

Corollary 3.4.42 $\Vdash^M A \iff \vDash^{KM} A$.

Corollary 3.4.43 $\Gamma \vDash_m^2 A$ implies $\Gamma \Vdash A$.

Proof: Identical to that of Corollaries 3.4.17 and 3.4.18. □

The adaptation of Definition 3.4.21 is also very simple, albeit not as immediate. We can no longer pick arbitrary 0-ary predicates P to use in structural vacuous rules or a arbitrary 1-ary Q to use in discursive rules, since we would be adding P and Q themselves to the natural domains. We must also figure out a way to add all the desired n-ary constants to the n-ary natural domains.

Since $\bot$ is in all 0-ary domains by definition (cf. Definition 3.4.35), we can define n-ary structural rules by using only $\bot$:

Definition 3.4.44 A n-ary vacuous $\bot$ rule is a rule conclusion $\bot$ from a sequence containing n occurrences of $\bot$.

The rules look as follows:

$$\frac{\bot}{\bot} \qquad \frac{\bot \quad \bot}{\bot} \qquad \frac{\bot \quad \bot \quad \bot}{\bot} \qquad (...)$$

They clearly add only $\bot$ to domains, so their inclusion in bases is not problematic. As for discursive rules, we pick predicate constants in the domain of the starting simplified model instead of picking a arbitrary 1-ary constants, so every n-ary constant used to introduce a individual constant is already a predicate constant that should be added to the n-ary domain. We also use the individual constants that should be added to the domain of individuals to produce vacuous rules with the predicate constants that should also be added to the n-ary domains (e.g. if $\alpha(k) = \{a\}$ and $\alpha^2(k) = \{P\}$ we can use the vacuous rule $[Paa/Paa]$ in S^k to add P and a to the domains).

We now adapt Definition 3.4.21 as follows:

Definition 3.4.45 For any second-order simplified minimal model K, a corresponding multibase M^K for it defined as follows:

1. Clauses 1, 2 and 3 are as in Definition 3.4.21;

4. If $k^n \leq k^m$, we add the n-ary $\bot$ rule $[\bot^1, \ ... \ , \bot^n/\bot]$ to S^k_m.

5. If $P_n \in \alpha^n(k)$ and $a^m \in \alpha(k)$ for $1 \leq m \leq n$, then $[P_n a^1 \ ... \ a^n / P_n a^1 \ ... \ a^n] \in S^k$;

6. Each S^k contains no rules other than those added by the above procedures.

Since the n-ary domain of each k is non-empty it follows that the multibase satisfies generalized existential import, as for every S^k there will be a rule $[P_n a^1 \ ... \ a^n / P_n a^1 \ ... \ a^n] \in S^k$ for each n.

Lemma 3.4.46 $k \leq k'$ if and only if $S^k \subseteq_M S^{k'}$.

Proof: Straightforward adaptation of Lemma 3.4.22. Notice that, for every n, $k \leq k'$ implies $\alpha(k) \subseteq \alpha(k')$ and $\alpha^n(k) \subseteq \alpha^n(k')$, so we still have that $[P_n a^1 \ ... \ a^n / P_n a^1 \ ... \ a^n] \in S^k$ implies $[P_n a^1 \ ... \ a^n / P_n a^1 \ ... \ a^n] \in S^{k'}$. □

Lemma 3.4.47 $\alpha(k) = N(S^k)$ and $\alpha^n(k) = N^n(S^k)$, for every n.

Proof: Straightforward adaptation of Lemma 3.4.23. □

Lemma 3.4.48 $\vDash^K_k A \iff \Vdash^{M^K}_{S^k} A$.

Proof: The proof for atoms, $\wedge$, $\vee$, $\rightarrow$, $\forall x$ and $\exists x$ is identical to that of Lemma 3.4.24. For $\forall X$ and $\exists X$ the proof can be obtained through a straightforward adaptation of the proofs for $\forall x$ and $\exists x$ in the same lemma. □

Corollary 3.4.49 $\Gamma \vDash^K A \iff \Gamma \Vdash^{M^K} A$.

Lemma 3.4.50 $\Gamma \Vdash A$ implies $\Gamma \vDash^2_m A$.

Proof: Identical to that of Lemma 3.4.26. □

Theorem 3.4.51 (Soundness and completeness) $\Gamma \vDash^2_m A \iff \Gamma \Vdash A$.

Theorem 3.4.52 (Focused soundness and completeness) $\Gamma \vDash^2_m A \iff \Gamma \Vdash^2_m A$.

Proof: The completeness proof in (VAN DALEN, 2013, pgs. 169-172) must first be adapted so that it also proves completeness for weak second-order logic with respect to weak second-order natural deduction. This can be done if we treat second-order quantification just like first-order quantification, so when constructing a prime theory Γ' for a set Γ of formulas we add witnessing n-ary constant P_n to the language for every n-ary quantification $\exists X_n(A)$ just like we add a individual constant a for each quantification $\exists x(A)$. In every other step of the proof we deal with second-order quantification in the same way we deal with first-order quantification. The result follows if we employ the same reasoning used in Theorems 3.4.27, 3.4.28, 3.2.25 and 3.2.30. □

Just as in the propositional and first-order cases, the result also holds for finite multibases and focused multibases.

As expected, there are few differences between first and second-order multibases, so weak second-order semantics is very close to the first-order semantics. Not all first-order multibases are second-order multibases, but only because satisfaction of existential import does not imply satisfaction of generalized existential import. Another minor but notable difference is that, even though multibases themselves may be finite, each base of a multibase satisfying generalized existential import is necessarily infinite. Only finitely many predicate constants occur in each atomic rule, so every finite set of rules contains only finitely many n-ary predicate constants; since we require all n-ary domains (with $n \geq 0$) to be non-empty, from the definition of natural domains it follows that the set of rules cannot be finite. This is not the case if we demand only the satisfaction of existential import. But those are clearly minor differences related to how we prevent natural domains from being empty, so they are still essentially the same. In fact, first and second-order multibases collapse into each other if domains are allowed to be empty.

We have argued before that weak second-order logic is interesting in its own right, but multibase semantics can also be extended to intermediate and strong second-order semantics. It is not immediately clear what restriction should be imposed on atomic derivability in order to obtain intermediate semantics, so this is left as a open question. Even though this is not ideal, it is still possible to obtain intermediate semantics by requiring second-order multibases to satisfy all instances of the comprehension schema.

In the case of strong second-order logic, the appropriate atomic restriction is immediately evident:

Definition 3.4.53 A second-order multibase M is a multibase for *strong second-order logic* if, for every $S \in M$, if R is a n-ary relation on the objects of $N(S)$ then there is a $P_n \in N^n(S)$ such that $\langle a^1, \ldots, a^n \rangle \in R$ if and only if $\vdash_S P_n(a^1 \ldots a^n)$.

If we consider only simplified second-order models it is clear that this restriction corresponds to Definition 2.4.10. A consequence relation for minimal strong second-order logic can then be defined by considering only multibases for strong second-order logic.

The significance of this last result lies in the fact that strong second-order logic has a proof-theoretic semantics in which validity is reduced to atomic derivability, *even though it has no syntactic system*. From this we conclude that validity in strong second-order logic can be reduced to derivability in collections of systems of natural deduction, even though it is not reducible to derivability in any single system of natural deduction.

3.4.5 Classical and intuitionistic predicate focused multibases

As in the case of propositional logic, from now on we deal only with focused multibases. We also fix the notation $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ for arbitrary atomic sentences.

The extension to first-order intuitionistic logic is straightforward, as we only need to adapt the definitions and proofs of Section 3.2.4 to the first-order case. On the other hand, the extension to classical logic is not immediate because, as commented soon after Definition 2.3.12, it is not enough to impose the restrictions of Definition 2.3.4 on atoms to obtain classical models from intuitionistic models.

This complication is avoided altogether by our way of defining classical multibases. When adapting Definitions 3.2.42 and 3.2.43 to first-order logic, we get the following:

Definition 3.4.54 A first-order intuitionistic focused multibase M is *classical* if, for every atomic sentence $\mathcal{A}$ of $\mathbb{P} \cup N(F)$ and every $S \in M$, either $\vdash_S \mathcal{A}$ or $\mathcal{A} \vdash_S \bot$.

This condition holds for every sentence in the language in every base. But this means that every sentence of the language must be used somehow in every base and, since every constant of the language occurs in some sentence, *every constant of the language is in the domain of every base*. We prove this as follows:

Lemma 3.4.55 If F is a first-order classical focused multibase, for every $S \in F$ we have $N(F) = N(S)$.

Proof: Since F is a classical multibase, for any $(n+1)$-ary predicate constant P, either $\vdash_S Pa^1 \dots a^n$ or $Pa^1 \dots a^n \vdash_S \bot$ for every atomic sentence $Pa^1 \dots a^n$ of $\mathbb{P} \cup N(F)$ and every $S \in F$. If $\vdash_S Pa^1 \dots a^n$, then there is a deduction with no premises and last formula $Pa^1 \dots a^n$, so there must be some rule with conclusion $Pa^1 \dots a^n$ in S, hence $a^m \in N(S)$ for every $1 \leq m \leq n$ due to the definition of natural domains.

If $Pa^1 \dots a^n \vdash_S \bot$, there must be a deduction with conclusion $\bot$ depending on $Pa^1 \dots a^n$ in S. Due to Definitions 2.2.5 and 2.5.8, for sets of atomic sentences Γ and Δ, if $\Gamma \vdash_S a$ then $\Gamma \cup \Delta \vdash_S a$, so it is not necessary for a sentence to occur in a deduction in order for the conclusion to depend on it. However, if there is a deduction showing $\Gamma \cup \Delta \vdash_S a$ which does not use any formula in Δ we may obtain a new deduction that omits the redundant formulas to show $\Gamma \vdash_S a$. As such, if the deduction did not use the formula $Pa^1 \dots a^n$ in any way, we could omit it in the deduction showing $Pa^1 \dots a^n \vdash_S \bot$ to obtain a deduction showing $\vdash_S \bot$, which would violate the consistency requirement. We then conclude that $Pa^1 \dots a^n$ cannot have been added vacuously to the deduction, so it must be used in some way by the rules. From the fact that the deduction showing $Pa^1 \dots a^n \vdash_S \bot$ uses a undischarged instance of $Pa^1 \dots a^n$ it follows that there is some rule in S with $Pa^1 \dots a^n$ as a premise, so once again $a^m \in N(S)$ for every $1 \leq m \leq n$ due to the definition of natural domains.

Hence every individual constant of $\mathbb{P} \cup N(F)$ which occurs in some atomic sentence must be included in $N(S)$. But from the definition of atomic sentence it follows that every individual constant of a language occurs in some atomic sentence of that language, so by the definition of natural domains every constant of the language $\mathbb{P} \cup N(F)$ must occur in $N(S)$, and since the only individual constants in $\mathbb{P} \cup N(F)$ are the elements of $N(F)$ (as the language $\mathbb{P}$ is pure), we conclude $a \in N(F)$ implies $a \in N(S)$. But by the definition of $N(F)$ we also have $a \in N(S)$ implies $A \in N(F)$, so $N(S) = N(F)$. □

This means that classical multibases have *constant natural domains*, which is sufficient to block the problem present in the model-theoretic definition.

Lemma 3.4.56 For all A and all S in a classical focused multibase F, either $\Vdash^F_{S'} A$ for all $S \subseteq S'$ or $\nVdash^F_{S'} A$ for all $S \subseteq_F S'$.

Proof: Atomic case, ($\wedge$), ($\vee$) and ($\rightarrow$): identical to Lemma 3.2.46.

($\forall x$): Assume $\Vdash^F_{S'} A[x/a]$ for all $a \in N(F)$ in all $S \subseteq_F S'$. By Lemma 3.4.55 we have $N(S') = N(F)$ for all such S', so for every $S \subseteq_F S'$ we have $\Vdash^F_{S'} A[x/a]$ for all $a \in N(S')$. Since for every $S' \subseteq_F S''$ we have $S \subseteq_F S''$ by transitivity and so $\Vdash^F_{S''} A[x/a]$ for all $a \in N(S'')$, we conclude $\Vdash^F_{S'} \forall x A$ for all $S \subseteq_F S'$. Now assume $\nVdash^F_{S'} A[x/a]$ for some $a \in N(F)$ in all $S \subseteq_F S'$. Since $N(F) = N(S')$ we have $\nVdash^F_{S'} A[x/a]$ for some $a \in N(S')$, so $\nVdash^F_{S'} \forall x A$, hence by arbitrariness of S' we conclude $\nVdash^F_{S'} \forall x A$ for all $S \subseteq_F S'$.

($\exists x$): Assume $\Vdash^F_{S'} A[x/a]$ for some $a \in N(F)$ in all $S \subseteq_F S'$. Since $N(F) = N(S')$ we have $\Vdash^F_{S'} A[x/a]$ for some $a \in N(S')$, so $\Vdash^F_{S'} \exists x A$, hence by arbitrariness of S' we conclude $\Vdash^F_{S'} \exists x A$ for all $S \subseteq_F S'$. Now assume $\nVdash^F_{S'} A[x/a]$ for all $a \in N(F)$ in all $S \subseteq_F S'$. Then for every $S \subseteq_F S'$ we have $\nVdash^F_{S'} A[x/a]$ for all $a \in N(S')$. Since for every $S' \subseteq_F S''$ we have $S \subseteq_F S''$ by transitivity and so $\nVdash^F_{S''} A[x/a]$ for all $a \in N(S'')$, we conclude $\nVdash^F_{S'} \exists x A$ for all $S \subseteq_F S'$. □

All proofs for classical propositional multibases in Section 3.2.4 can be extended to first-order multibases if we replace every use of Lemma 3.2.46 by a use of Lemma 3.4.56.

The extension to second-order intuitionistic multibases is also straightforward, and the extension to classical logic can be obtained by promoting two small adaptations on the previous proofs. Lemma 3.4.55 must be adapted so that deduction with 0-ary predicates are also considered, hence all n-ary predicate constants can be shown to be in the n-ary natural domains through the same strategy. Notice that a single occurrence of $\bot$ is already a deduction showing $\bot \vdash_S \bot$ in any S, so our argument does not show that $\bot$ occurs in the premise or the conclusion of some rule – but this is not problematic because satisfaction of generalized existential import already guarantees that $\bot$ occurs in some rule. Lemma 3.4.56 must be adapted in order to deal with second-order quantification, but since we can deal with second-order quantification in the same way we have dealt whit first-order quantification the adaptation is straightforward. Once those adaptations have been made, all results in Section 3.2.4 can be extended to second-order classical logic.

3.5 Results for predicate generalized S-validity

3.5.1 Extension of propositional results

All of the results in Sections 3.3.1 and 3.3.2 can be extended to first-order minimal focused multibases, with the exception of the proof of Export. This is so because $\langle\emptyset\rangle$ does not satisfy existential import and is thus not a first-order multibase, hence we cannot use Lemma 3.3.18. For reasons that will soon become clear, there is also a additional problem concerning uses of the isomorphism lemma that cannot be corrected, so Export does not hold in general. But a limited version of Export is still available.

We start by extending monotonicity and isomorphism to first-order multibases. The statement of monotonicity is still the same, but notice that isomorphism now requires identity of domains:

Lemma 3.5.1 (Monotonicity) For any focused multibase F and any $S \in F$, $\Vdash_S^F A$ and $S \subseteq_F S'$ implies $\Vdash_{S'}^F A$.

Proof:

1. Atomic case, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$: identical to Lemma 3.3.2.
2. $(\forall x)$: Assume $\Vdash_S^F \forall x A$. Then for all $S \subseteq_F S'$ we have $\Vdash_{S'}^F A[x/a]$ for all $a \in N(S')$. Pick any $S' \subseteq_M S''$. By transitivity of $\subseteq_F$ we have $S \subseteq_F S''$ and so $\Vdash_{S''}^M [x/a]$ for all $a \in N(S'')$, so by arbitrariness of S'' we have $\Vdash_{S'}^F \forall x A$.
3. $(\exists x)$: Assume $\Vdash_S^F \exists x A$. Then $\Vdash_S^F A[x/a]$ for some $a \in N(S)$. Fix the a. Induction hypothesis: for every $S \subseteq_F S'$ we have $\Vdash_{S'}^F A[x/a]$. But by Lemma 3.4.13 we have $a \in N(S')$, so $\Vdash_{S'}^F \exists x A$. □

Lemma 3.5.2 (Isomorphism) Let M and M' be multibases. Let $Q = \langle S_1, S_2, ...\rangle$ be a subsequence of M such that $S_n \subseteq_M S$ implies $S \in Q$ for all $(n \geq 1)$, and $Q' = \langle S_1^{'}, S_2^{'}, ...\rangle$ a subsequence of M' such that $S_n^{'} \subseteq_{M'} S'$ implies $S' \in Q'$ for all $(n \geq 1)$. If it holds that $N(S_n) = N(S_n^{'})$, $\vdash_{S_n} \mathcal{A}$ if and only if $\vdash_{S_n^{'}} \mathcal{A}$ and $S_n \subseteq_M S_m$ if and only if $S_n^{'} \subseteq_{M'} S_m^{'}$ for all $(n \geq 1)$ and $(m \geq 1)$, then $\Vdash_{S_n}^M A$ if and only if $\Vdash_{S_n^{'}}^{M'} A$, and also $\Gamma \Vdash_{S_n}^M A$ if and only if $\Gamma \Vdash_{S_n^{'}}^M A$.

Proof:

1. Atomic case, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$: identical to Lemma 3.3.3.

2. $(\forall x)$: Assume $\Vdash^{M}_{S_n} \forall x A$. Then for all $S_n \subseteq_M S$ we have $\Vdash^{M}_{S} A[x/a]$ for all $a \in N(S)$, and since $S_n \subseteq_M S$ implies $S \in Q$ we have that all such S are $S_m \in Q$. Induction hypothesis: if $S_n \subseteq_M S_m$ then $\Vdash^{M'}_{S'_m} A[x/a]$ for all $a \in N(S_m)$. But $N(S_m) = N(S'_m)$, so also $\Vdash^{M'}_{S'_m} A[x/a]$ for all $a \in N(S'_m)$. But for every $S'_n \subseteq_{M'} S'$ we have that S' is some $S'_m \in Q'$, and since $S_n \subseteq_M S_m$ if and only if $S'_n \subseteq_{M'} S'_m$ we have $\Vdash^{M'}_{S'_m} A[x/a]$ for all $a \in N(S'_m)$ in any $S'_n \subseteq_{M'} S'_m$, so also $\Vdash^{M'}_{S'} A[x/a]$ for all $a \in N(S')$ in any $S'_n \subseteq_{M'} S'$, hence $\Vdash^{M'}_{S'_n} \forall x A$. The converse can be proved in a similar fashion.

3. $(\exists x)$: Assume $\Vdash^{M}_{S_n} \exists x A$. Then $\Vdash^{M}_{S_n} A[x/a]$ for some $a \in N(S_n)$. Induction hypothesis: $\Vdash^{M'}_{S'_n} A[x/a]$ for some $a \in N(S_n)$. But $N(S_n) = N(S'_n)$, so also $\Vdash^{M'}_{S'_n} A[x/a]$ for some $a \in N(S'_n)$, hence $\Vdash^{M'}_{S'_n} \exists x A$. The converse can be proved in a similar fashion.

To finish the proof we just repeat the final step of Lemma 3.3.3. □

The additional requirement of the isomorphism lemma is mostly unproblematic because, with the exception of Lemma 3.3.17, in all proofs of Section 3.3 we apply it only in contexts that guarantee equality of domains, provided only vacuous structural rules have been used to order bases. However, the new requirement of the isomorphism lemma is indeed problematic in a particular step of the proof of Export: there is no guarantee that $N(S^2) = N(S^2_v \cup R)$, as the R could be adding new constants to the natural domain. This cannot be fixed, as shown by the following:

Theorem 3.5.3 $\Gamma, S^* \Vdash^1_m A$ implies $\Gamma \Vdash_S A$, provided only first-order production multibases are admitted in the semantics.

Proof: Assume $\Gamma, S^* \Vdash^1_m A$. Let F be any focused multibase for S. By Theorem 3.3.7 and through use of the standard mapping we have $\Vdash_S R^*$, thus $\Vdash^F R^*$ and then $\Vdash^F_{S'} R^*$ in every $S' \in F$ and for every $R \in S$, so by Definition 3.3.13 also $\Vdash^F_{S'} S^*$ in all such S'. Now assume $\Vdash^F_{S''} \Gamma$ for some $S'' \in F$. Then clearly $\Vdash^F_{S''} \Gamma \cup S^*$. From our assumption we have $\Gamma, S^* \Vdash_S A$ and thus $\Gamma, S^* \Vdash^F A$, so also $\Gamma, S^* \Vdash^F_{S''} A$, hence from $\Vdash^F_{S''} \Gamma \cup S^*$ we obtain $\Vdash^F_{S''} A$. But this was a arbitrary S'' such that $\Vdash^F_{S''} \Gamma$, hence for all $S' \in F$ we can conclude $\Gamma \Vdash^F_{S'} A$, hence from arbitrariness of S' we conclude $\Gamma \Vdash^F A$, whence from arbitrariness of F we conclude $\Gamma \Vdash_S A$. □

Theorem 3.5.4 It is not the case that $\Gamma, S^* \Vdash^1_m A$ if and only if $\Gamma, \Vdash_S A$ implies $\Gamma, S^* \Vdash^1_m A$, even if only production multibases are admited in the semantics.

Proof: For a counterexamples, let $S = \{[Pa/Pa]\}$. Let F be any focused multibase for S. Let $S' \in F$, and assume $\Vdash^F_{S''} \forall x Px$ for some $S' \subseteq_F S''$. We have $a \in N(S'')$ by Lemma

3.4.13, so in particular $\Vdash^F_{S''} Pa$. But this is a arbitrary $S' \subseteq_F S''$, so we conclude $\forall x Px \Vdash^F_{S'} Pa$. S' is also arbitrary, so $\forall x Px \Vdash^F_{S'} Pa$ holds for all $S' \in F$, hence $\forall x \Vdash^F Pa$, and since F is a arbitrary multibase focused on S we have $\forall x Px \Vdash_S Pa$.

The standard mapping yields $[Pa/Pa] = (Pa \rightarrow Pa)$, so we must show $(\forall x Px, Pa \rightarrow Pa \nVdash Pa)$. Let $F' = \langle S^1 = \{[/Pb]\},\ S^2 = \{[/Pa], [/Pb]\}\rangle$. It is easy to see that $(\Vdash^{F'}_{S^1} \forall x Px, Pa \rightarrow Pa)$ but $(\nVdash^{F'}_{S^1} Pa)$, so $(\forall x Px, Pa \rightarrow Pa \nVdash^{F'}_{S^1} Pa)$, hence $(\forall x Px, Pa \rightarrow Pa \nVdash^{F'} Pa)$, hence $(\forall x Px, Pa \rightarrow Pa \nVdash_{S^1} Pa)$, and we conclude $(\forall x Px, Pa \rightarrow Pa \nVdash Pa)$. □

Corollary 3.5.5 Export does not hold in general for first-order multibase semantics.

This problem is caused specifically by interactions between the universal quantifier and bases, so a restricted version of Export can still be proved. We started by proving a weakened version of isomorphism, in which domains are not required to be equal but formulas are not allowed to contain universal quantifiers as subformulas.

First we prove two preparatory lemmas on substitution, which are equivalent to Theorem 3.5.6. and Definition 3.3.10 of (VAN DALEN, 2013):

Lemma 3.5.6 If $\Vdash^F_S \exists x A$ and y does not occur in A, then $\Vdash^F_S \exists x A \iff \Vdash^F_S \exists y(A[x/y])$.

Corollary 3.5.7 Every formula A is equivalent to a formula A^* such that no variable x occuring in A^* is bound by two different quantifiers.

Proof: Let $\Vdash^F_S \exists x A$. The formula $A[x/y]$ consists in the original A with y at every place in which x was, and since y does not occur in A it only occurs in $A[x/y]$ on places in which x was. As such, we have $\Vdash^F_S \exists x A$ implies $\Vdash^F_S A[x/a]$ implies $\Vdash^F_S A[x/y][y/a]$ implies $\Vdash^F_S \exists y A[x/y]$. For the converse, let $\Vdash^F_S \exists y A[x/y]$. Then $\Vdash^F_S A[x/y][y/a]$. But then instead of replacing x by y and y by a we can directly replace x by a, so $\Vdash^F_S A[x/y][y/a]$ implies $\Vdash^F_S A[x/a]$, hence $\Vdash^F_S \exists x A$.

The corollary follows from the fact that every subformula $\exists x B$ of A can be replaced by the equivalent formula $\exists y(B[x/y])$ for y not in B, and if we do this using a distinct y for every distinct quantifier the result follows. □

Lemma 3.5.8 The following hold:

1. $A \wedge B[x/a] = A[x/a] \wedge B[x/a]$;
2. $A \vee B[x/a] = A[x/a] \vee B[x/a]$;
3. $A \rightarrow B[x/a] = A[x/a] \rightarrow B[x/a]$;
4. If $x = y$ then $\exists y B[x/a] = \exists y B$, and if $x \neq y$ then $\exists y B[x/a] = \exists y(B[x/a])$.

Proof: The first three items follow immediately from our definition of substitution. As for the fourth, notice that if $x = y$ then there is no free occurrences of x in B, so $\exists y B[x/a] = \exists y B$ by our definition of substitution. If $x \neq y$ then $\exists y B[x/a]$ is obtained by replacing all occurences of x in B by a, which is the same formula as $\exists y(B[x/a])$. $\square$

Lemma 3.5.8 could be a definition instead of a Lemma, but this makes no difference.

Lemma 3.5.9 (Weakened isomorphism) Let M and M' be multibases. Let $Q = \langle S_1, S_2, ...\rangle$ be a subsequence of M such that $S_n \subseteq_M S$ implies $S \in Q$ for all $(n \geq 1)$, and $Q' = \langle S_1^{'}, S_2^{'}, ...\rangle$ a subsequence of M' such that $S_n^{'} \subseteq_{M'} S'$ implies $S' \in Q'$ for all $(n \geq 1)$. If it holds that $\vdash_{S_n} \mathcal{A}$ if and only if $\vdash_{S_n^{'}} \mathcal{A}$ and $S_n \subseteq_M S_m$ if and only if $S_n^{'} \subseteq_{M'} S_m^{'}$ for all $(n \geq 1)$ and $(m \geq 1)$, then $\Vdash_{S_n}^{M} A$ if and only if $\Vdash_{S_n^{'}}^{M'} A$, and also $\Gamma \Vdash_{S_n}^{M} A$ if and only if $\Gamma \Vdash_{S_n^{'}}^{M} A$, provided no formula of $\Gamma \cup A$ has some formula of shape $\forall x B$ as its subformula.

Proof: We first prove by induction that, for any multibase satisfying those conditions, $a \in N(S_n)$ and $\Vdash_{S_n}^{M} A[x/a]$ if and only if $a \in N(S_n^{'})$ and $\Vdash_{S_n^{'}}^{M'} A[x/a]$ for all S_n and all a, provided $A[x/a]$ does not have universal quantifications as subformulas. We assume that Lemma 3.5.6 and Corollary 3.5.7 have been used to substitute every variable bound by two distinct quantifiers on the same formula, so no subformula of a formula $\exists x A$ has a subformula of shape $\exists x B$ for some B. The induction is a straightforward adaptation of proof steps of Lemmas 3.3.3 and 3.5.2, but we write them again in order to show how substitutions factor in.

1. Atomic case: if $a \in N(S_n)$ and $\Vdash_{S_n}^{F} A[x/a]$, then $\vdash_{S_n} A[x/a]$ and so $\vdash_{S_n} A[x/a]$ by construction, so $\Vdash_{S_n^{'}}^{F} A[x/a]$ by construction, and since there is a deduction of $A[x/a]$ in $S_n^{'}$ then there must be a rule concluding $A[x/a]$ in $S_n^{'}$, so $a \in N(S_n^{'})$. Proof of the converse is similar.

2. $(A[x/a] = (B \wedge C)[x/a])$. Since $\Vdash_{S_n}^{M} (B \wedge C)[x/a]$, we have $\Vdash_{S_n}^{M} B[x/a]$ and $\Vdash_{S_n}^{M} C[x/a]$. Induction hypothesis: $\Vdash_{S_n^{'}}^{M} B[x/a]$ and $\Vdash_{S_n^{'}}^{M} C[x/a]$, with $a \in N(S_n^{'})$, hence $\Vdash_{S_n^{'}}^{M} (B \wedge C)[x/a]$ for $a \in N(S_n^{'})$. Proof of the converse is similar.

3. $(A[x/a] = (B \vee C)[x/a])$. Since $\Vdash_{S_n}^{M} (B \vee C)[x/a]$, we have $\Vdash_{S_n}^{M} B[x/a]$ or $\Vdash_{S_n}^{M} C[x/a]$. Induction hypothesis: $\Vdash_{S_n^{'}}^{M} B[x/a]$ or $\Vdash_{S_n^{'}}^{M} C[x/a]$ with $a \in N(S_n^{'})$, hence in both cases we have $\Vdash_{S_n^{'}}^{M} (B \vee C)[x/a]$ for $a \in N(S_n^{'})$. Proof of the converse is similar.

4. $(B[x/a] = (B \rightarrow C)[x/a])$. Since $\Vdash_{S_n}^{M} (B \rightarrow C)[x/a]$, we have $B[x/a] \Vdash_{S_n}^{M} C[x/a]$, so for every $S^n \subseteq_M S$ we have $\Vdash_{S}^{M} B[x/a]$ implies $\Vdash_{S}^{M} C[x/a]$. But $S^n \subseteq_M S$ implies that S is some $S_m \in Q'$, so $\Vdash_{S_m}^{M} B[x/a]$ implies $\Vdash_{S_m}^{M} C[x/a]$ for all $S_n \subseteq_M S_m$. Induction hypothesis: $\Vdash_{S_m}^{M} B[x/a]$ implies $\Vdash_{S_m}^{M} C[x/a]$ with $a \in N(S_n^{'})$ for all $S_n \subseteq_M S_m^{'}$. But $S_n \subseteq_M S_m$ if and only if $S_n^{'} \subseteq_M S_m^{'}$ and $S_n^{'} \subseteq_M S'$ if and only if S' is some $S_m^{'} \in Q'$, so $\Vdash_{S'}^{M} B[x/a]$ implies $\Vdash_{S'}^{M} C[x/a]$ for all $S_n^{'} \subseteq_M S'$, hence $\Vdash_{S_n^{'}}^{M} (B \rightarrow C)[x/a]$ for $a \in N(S_n^{'})$. Proof of the converse is similar.

5. $(A[x/a] = (\exists yB)[x/a])$. Our previous applications of Lemma 3.5.6 and Corollary 3.5.7 guarantee that $x \neq y$. Assume $\Vdash^{M}_{S_n} (\exists yB)[x/a]$. Then $\Vdash^{M}_{S_n} \exists y(B[x/a])$. By the semantic clause we have $\Vdash^{M}_{S_n} B[x/a][y/b]$ for some $b \in N(S_n)$. Induction hypothesis: $\Vdash^{M'}_{S'_n} B[x/a][y/b]$ and $b \in N(S'_n)$, hence we have $\Vdash^{M'}_{S'_n} (\exists yB[x/a])$, hence $\Vdash^{M'}_{S'_n} (\exists yB)[x/a]$. Proof of the converse is similar.

Since no formula has $\forall xB$ as a subformula and validity is defined inductively we do not need to treat it in the induction steps.

Now for the remaining steps of the proof:

1. Atomic case, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$: identical to Lemma 3.3.3.
2. $(\exists x)$: Assume $\Vdash_{S_n} \exists xA$. Then $\Vdash_{S_n} A[x/a]$ for some $a \in N(S_n)$. Induction hypothesis: $\Vdash_{S'_n} A[x/a]$ for some $a \in N(S_n)$. By the previous result we have $a \in N(S'_n)$, so we conclude $\Vdash_{S'_n} \exists xA$. Proof of the converse is similar.

To finish the proof we just repeat the final step of Lemma 3.3.3. $\square$

The intuition behind weakened isomorphism is that by guaranteeing equality of derivability we also guarantee equality of the "witnessing constants" for existential quantifiers, so it is not necessary to require domains to be equal.

Now we are finally ready to prove the restricted Export:

Theorem 3.5.10 $\Gamma, S^* \Vdash^1_m A$ if and only if $\Gamma \Vdash_S A$, provided only first-order production multibases are admitted in the semantics and no formula in $\Gamma \cup A$ has universal quantifiers as subformulas.

Proof:

$(\Rightarrow)$ is a particular case of Theorem 3.5.3. For $(\Leftarrow)$, assume $\Gamma \Vdash_S A$. Let F be any multibase focused on any base, and assume $\Vdash^{F}_{S^1} \Gamma \cup S^*$ for some $S^1 \in F$. We make the same constructions used in Lemma 3.3.17, but instead of building sets $S^2_v \cup R$ we build sets $S^2_v \cup S$. We then create a multibase F^1 focused on S such that $(S^2_v \cup S) \in F^1$ if and only if $S' \subseteq_F S^2$ for all bases different from S. We let F^2 be the multibase obtained by deleting the first S and putting $S^1_v \cup S$ at the start of the sequence, so from isomorphism (possibly after a reordering of F^2) we have $\Gamma \Vdash^{F}_{S^2_v \cup S} A$ for every $(S^2_v \cup S) \in F$.

Let Π be a deduction in a arbitrary $(S^2_v \cup S) \in F'^2$ showing $\vdash_{S^2_v \cup S} \mathcal{A}$. By removing all vacuous rules from Π we obtain a deduction Π' showing $\vdash_{S^2 \cup S} \mathcal{A}$. If there is a $[/\mathcal{B}] \in S$ then $\mathcal{B} \in S^*$, so there must be a deduction in S^1 showing $\vdash_{S^1} \mathcal{B}$ and so also $\vdash_{S^2} \mathcal{B}$, hence all axiomatic rules used in Π' can be replaced as follows:

$$\dfrac{\overline{\mathcal{B}_1} \quad \dots \quad \overline{\mathcal{B}_n}}{\begin{matrix}\Pi^j \\ \mathcal{A}\end{matrix}} \quad \text{is transformed into} \quad \begin{matrix} \Pi^1 & & \Pi^n \\ \mathcal{B}_1 & \dots & \mathcal{B}_n \\ & \Pi^j & \\ & \mathcal{A} & \end{matrix}$$

After replacing every rule we obtain a deduction Π_n that uses no axiomatic rules, n being the number of production rules not in S^2 that are used in the deduction. If $n > 0$, we can use the procedure shown in the proof of Lemma 3.3.17 to remove the uppermost rule and obtain a deduction Π_{n-1}, and the process can be reiterated until $n = 0$, which is a deduction showing $\vdash_{S^2} \mathcal{A}$. Since $\mathcal{A}$ was a arbitrary atom and $\vdash_{S^2} \mathcal{A}$ also implies $\vdash_{S^2_v \cup S} \mathcal{A}$ we conclude $\vdash_{S^2} \mathcal{A}$ if and only if $\vdash_{S^2_v \cup S} \mathcal{A}$. Unlike in Theorem 3.3.17, we cannot use isomorphism to conclude the desired results because the domains of S^2 and $(S^2_v \cup S)$ might not be equal. But since no formulas in $\Gamma \cup A$ have universal quantifiers as their subformulas we can use apply the weakened isomorphism lemma to F and F^2, hence $\Gamma, S^* \Vdash^F_{S^2} A$ for all $S^1 \subseteq_F S^2$. Since $\Vdash^F_{S^1} \Gamma \cup S^*$ by monotonicity $\Vdash^F_{S^2} \Gamma \cup S^*$ for all such S^2, so for all S^2 we have $\Vdash^F_{S^2} A$. But this is an arbitrary $S^1 \in F$ such that $\Vdash^F_{S^1} \Gamma \cup S^*$, hence $\Gamma, S^* \Vdash^F_{S^3} A$ for all $S^3 \in F$. Since S^3 is arbitrary we conclude $\Gamma \Vdash^F A$, and since F is a arbitrary multibase focused on a arbitrary base we conclude $\Gamma, S^* \Vdash^1_m A$. □

So even though Export does not hold in general, this limited version still holds in predicate logic. All other results for propositional logic extend to predicate logic without problems, so this is our only real loss.

The treatment of second-order is similar, but there is once again an additional complication. Unlike in the propositional and first-order cases, we cannot freely add new 0-ary constants in the language to produce new structural vacuous rules without impacting quantification over 0-ary predicates (this was the very reason that led us to use n-ary vacuous $\bot$ rules instead of the usual structural rules). To remedy this, instead of adding a collection of 0-ary constants we add a single special constant $\bot^*$, to be used as a replacement of $\bot$ whenever there are no n-ary $\bot$ rules available. We define n-ary $\bot^*$ vacuous rules as rules concluding $\bot^*$ from a sequence with n occurrences of $\bot^*$. Additionally – and this is the important part – we add a rule $[/\bot^*]$ to a base S that is being ordered whenever $\vdash_S \bot$, so that we have $\vdash_S \bot$ if and only if $\Vdash_S \bot^*$ on every base ordered through the use of vacuous $\bot^*$ rules.

This solves the problem because we always start by defining validity or consequence for formulas $\Gamma \cup A$ occuring in the original language, so $\bot^*$ cannot occur in $\Gamma \cup A$. It is, however, a subformula of formulas with 0-ary quantifications, but since $\vdash_S \bot$ if and only if $\Vdash_S \bot^*$ by design we can always replace $\bot^*$ by $\bot$ in quantifications.

We exemplify this by adapting the isomorphism lemma:

Lemma 3.5.11 (Adapted isomorphism) Let M and M' be two multibases. Let $Q = \langle S_1, S_2, ...\rangle$ be a subsequence of M such that $S_n \subseteq_M S$ implies $S \in Q$ for all $(n \geq 1)$, and $Q' = \langle S'_1, S'_2, ...\rangle$ a subsequence of M' such that $S'_n \subseteq_{M'} S'$ implies $S' \in Q'$ for all $(n \geq 1)$. Let $\bot^*$ be a 0-ary predicate constant not in $\mathbb{L}((N(M))$. If it holds that:

1. $\Vdash_{S'_n} \bot$ if and only if $\Vdash_{S'_n} \bot^*$ for all $S'_n \in Q'$;

2. $N^0(S_n) = (N^0(S'_n) \cup \bot^*))$;

3. $N(S_n) = N(S_n^{'})$;

4. $N^m(S_n) = N^m(S_n^{'})$ for all $m > 0$;

5. $\vdash_{S_n} \mathcal{A}$ if and only if $\vdash_{S_n^{'}} \mathcal{A}$ for $\mathcal{A} \neq \bot^*$;

6. $S_n \subseteq_M S_m$ if and only if $S_n^{'} \subseteq_{M'} S_m^{'}$ for all $(n \geq 1)$ and $(m \geq 1)$;

7. $\Gamma \cup A$ is a set of formulas in the language $\mathbb{L}(N(M))$;

Then $\Vdash_{S_n}^{M} A$ if and only if $\Vdash_{S_n^{'}}^{M'} A$, and also $\Gamma \Vdash_{S_n}^{M} A$ if and only if $\Gamma \Vdash_{S_n^{'}}^{M} A$.

Proof:

1. Atomic case: since $\mathcal{A} \in \mathbb{L}(N(M))$ and $\bot^* \notin \mathbb{L}(N(M))$ we have $\mathcal{A} \neq \bot^*$, and the result follows from the fact that $\vdash_{S_n} \mathcal{A}$ if and only if $\vdash_{S_n^{'}} \mathcal{A}$ for $\mathcal{A} \neq \bot^*$.

2. $(A \wedge B)$, $(A \vee B)$and $(A \rightarrow B)$: identical to Lemma 3.3.3.

3. $(\forall x)$ and $(\exists x)$: identical to Lemma 3.5.2.

4. $(\forall X_0)$: We invert the usual order and start by proving $(\Leftarrow)$.

   Assume $\Vdash_{S_n^{'}}^{M'} \forall X_0 A$. Then $S_n^{'} \subseteq_{M'} S'$ implies $\Vdash_{S'}^{M'} A[X_0/P]$ for all $P \in N^0(S')$, and since $S_n^{'} \subseteq_{M'} S'$ implies $S' \in Q'$ all such S' are $S_m^{'} \in Q'$. Induction hypothesis: if $S_n^{'} \subseteq_{M'} S_m^{'}$ then $\Vdash_{S_m}^{M} A[X_0/P]$ for all $P \in N^0(S_m^{'})$.

   For all $P \in N^0(S_m)$ we have $P \neq \bot^*$ implies $P \in N^0(S_m^{'})$, so $\Vdash_{S_m}^{M'} A[X_0/P]$ holds for all $P \neq \bot^*$. This includes $\bot$, so $\Vdash_{S_m}^{M} A[X_0/\bot]$ holds. But $\Vdash_{S_m}^{M} A[X_0/\bot]$ if and only if $\Vdash_{S_m}^{M} A[X_0/\bot^*]$ (easily proved by a straightforward induction), hence $\Vdash_{S_m}^{M} A[X_0/\bot^*]$ and so $\Vdash_{S_m}^{M} A[X_0/P]$ for all $P \in N^0(S_n)$.

   For every $S_n \subseteq_M S$ we have that S is some $S_m \in Q$, and since $S_n^{'} \subseteq_{M'} S_m^{'}$ if and only if $S_n \subseteq_M S_m$ we have $\Vdash_{S_m}^{M} A[X_0/P]$ for all $P \in N^0(S_m)$ in any $S_n \subseteq_M S_m$, and so also $\Vdash_{S}^{M} A[X_0/P]$ for all $P \in N^0(S)$ in any $S_n \subseteq_M S$, hence $\Vdash_{S_n}^{M} \forall X_0 A$. The converse can be proved in a similar fashion; no special treatment for $\bot^*$ is needed in the inductive step because $\Vdash_{S_m^{'}}^{M'} A[X_0/P]$ for all $P \in N^0(S_m)$ already implies $\Vdash_{S_m^{'}}^{M'} A[X_0/P]$ for all $P \in N^0(S_m^{'})$.

5. $[\exists X_0]$: Assume $\Vdash_{S_n}^{M} \exists X_0 A$. Then $\Vdash_{S_n}^{M} A[X_0/P]$ for some $P \in N^0(S_n)$. Induction hypothesis: $\Vdash_{S_n^{'}}^{M'} A[X_0/P]$ for some $P \in N^0(S_n)$. But $N^0(S_n) \subseteq N^0(S_n^{'})$, so also $\Vdash_{S_n^{'}}^{M'} A[X_0/P]$ for some $P \in N(S_n^{'})$, hence $\Vdash_{S_n^{'}}^{M'} \exists X_0 A$. For the converse, assume $\Vdash_{S_n^{'}}^{M} \exists X_0 A$. Then $\Vdash_{S_n^{'}}^{M} A[X_0/P]$ for some $P \in N(S_n^{'})$. If $P \neq \bot^*$ then $P \in N^0(S_n)$, hence $\Vdash_{S_n^{'}}^{M} A[X_0/P]$ for some $P \in N(S_n)$. If $P = \bot^*$ then $\Vdash_{S_m^{'}}^{M'} A[X_0/\bot]$ if and only if $\Vdash_{S_m^{'}}^{M'} A[X_0/\bot^*]$, so $\Vdash_{S_n^{'}}^{M} A[X_0/\bot]$, thus $\Vdash_{S_n^{'}}^{M} A[X_0/P]$ for some $P \in N^0(S_n)$ (namely, $\bot$). In both cases we can conclude $\Vdash_{S_n}^{M} \exists X_0 A$, as desired.

6. $[\forall X_n]$ and $[\exists X_n]$, for $n > 0$: straightforward adaptation of the proof for $\forall x$ and $\exists x$ in Lemma 3.5.2.

To finish the proof we just repeat the final step of Lemma 3.3.3. □

The inclusion of $\bot^*$ is not problematic because it does not occur in the formulas $\Gamma \cup A$ of the original consequence and because it can always be replaced by $\bot$ on inductive steps. Adaptation of all remaining proofs for second-order logic is straightforward, so we only need to be mindful about making derivability for $\bot$ and $\bot^*$ coincide in all bases. This is also only necessary in minimal multibases, as in classical and intuitionistic multibases the consistency requirement makes it so that we only need to add vacuous rules (since then we will have both $\nvdash_S \bot$ and $\nvdash_S \bot^*$ in all bases).

As for intuitionistic predicate logic, no results of Section 3.3.1, 3.3.2 3.3.3 needs any adaptation besides those shown above, so they also hold for predicate logic. The only extra adaptations needed for classical logic are presented in Section 3.4.5, so the results for propositional classical multibases also hold for predicate classical multibases.

3.5.2 New results

The results at the end of Section 3.3.1 also have counterparts in predicate logic. As usual, we start proving the results for first-order multibases:

Theorem 3.5.12 The following equivalences holds:

1. $\Vdash_S \forall x A \iff \Vdash_{S'} A[x/a]$ for all $a \in N(S')$ in all $S \subseteq S'$;
2. $\Vdash_S \exists x A \iff \Vdash_S A[x/a]$ for some $a \in N(S)$.

Proof:

1. $(\forall x)$: Assume $\Vdash_S \forall x A$, and let S' be an arbitrary base with $S \subseteq S'$. Let F' be a arbitrary multibase focused on S'. For any $S^k \in F'$, define $S^k_{F''}$ as the base obtained from S^k by adding a vacuous structural rule $[P^j/P^j]$ to S^k whenever $S^j \subseteq_{F'} S^k$. Define F'' as multibase focused on the base $S'_{F''}$ (obtained from S') such that $S^k \in F'$ if and only if $S^k_{F''} \in F''$.

   Now let F be the focused multibase $F = \langle S, S'_{F''}, ... \rangle$, obtained by putting S at the start of F''. Since this is a multibase focused on S, the original assumption yields $\Vdash^F \forall x A$, so $\Vdash^F_{S''} \forall x A$ for all $S'' \in F$. In particular, $\Vdash^F_{S'_{F''}} \forall x A$ and also $\Vdash^F_{S^m_{F''}} A[x/a]$ for all $a \in N(S^m_{F''})$ in all $S'_F \subseteq_{F''} S^m_{F''}$. By putting $S'_{F''} = S^m_{F''}$ we have $\Vdash^F_{S'_{F''}} A[x/a]$ for all $a \in N(S'_{F''})$, and by monotonicity also $\Vdash^{F}_{S^m_{F''}} A[x/a]$ for all $a \in N(S'_{F''})$ in all $S'_{F''} \subseteq_F S^m_{F''}$. By Lemma 3.5.2 we have $\Vdash^{F'}_{S^k} A[x/a]$ for every $a \in N(S')$ in every $S^k \in F'$, so $\Vdash^{F'} A[x/a]$ for every $a \in N(S')$. But this is an arbitrary F' focused on S', so $\Vdash_{S'} A[x/a]$ for all $a \in N(S')$. The base S' is also an arbitrary base with $S \subseteq S'$, so $\Vdash_{S'} A[x/a]$ for all $a \in N(S')$ in all $S \subseteq S'$, as desired.

For the converse, assume $\Vdash_{S'} A[x/a]$ for all $a \in N(S')$ in all $S \subseteq S'$. Let F be a multibase focused on S, and pick any $S'' \in F$. Let F'' be the multibase focused on S'' such that $S''' \in F''$ if and only if $S'' \subseteq_F S'''$. Since F'' is a focused multibase for S'' and $S \subseteq S''$ we have $\Vdash_{S''} A[x/a]$ for all $a \in N(S'')$, hence $\Vdash^{F''} A[x/a]$ for all $a \in N(S'')$ and in particular $\Vdash^{F''}_{S''} A[x/a]$ for all $a \in N(S'')$, which by isomorphism yields $\Vdash^{F}_{S''} A[x/a]$ for all $a \in N(S'')$. But S'' is a arbitrary base in F, so for all $S'' \in F$ we have $\Vdash^{F}_{S''} A[x/a]$ for all $a \in N(S'')$. This can be used together with the clause for universal quantification to conclude $\Vdash^{F}_{S''} \forall x A$ for all $S'' \in F$, hence $\Vdash^{F} \forall x A$, whence $\Vdash_S \forall x A$ by arbitrariness of F.

2. $(\exists x)$: The proof is a straightforward adaptation of the inductive step for disjunction in Theorem 3.3.4. We omit some details in the second part for the sake of simplicity.

   Assume $\Vdash_S A[x/a]$ for some $a \in N(S)$. Then $\Vdash^{F} A[x/a]$ for all F focused on S and $\Vdash^{F}_{S'} A[x/a]$ for all $S' \in F$. Since $a \in N(S)$ and F is focused on S, by Lemma 3.4.13 we have $a \in N(S')$ for all $S' \in F$, hence $\Vdash^{F}_{S'} \exists x A$, whence $\Vdash^{F} \exists x A$ by arbitrariness of S' and $\Vdash_S \exists x A$ by arbitrariness of F.

   For the converse, assume $\nVdash_S A[x/a]$ for all $a \in N(S)$ but still $\Vdash_S \exists x A$. For every $b \in N(S)$ there must be a multibase F^b focused on S such that $\nVdash^{F^b} A[x/b]$. For each F^b we create a multibase F^b_v by including vacuous rules in the bases of F^b just like in the inductive step for disjuction, and after this is done we put all bases in the multibases F^x_v (for $x \in N(S)$) in a single multibase F focused on S. Since this is a multibase focused on S, from the assumption that $\Vdash_S \exists x A$ we have $\Vdash^{F} \exists x A$, hence $\Vdash^{F}_{S} \exists x A$, whence $\Vdash^{F}_{S} A[x/c]$ for some $c \in N(S)$. By monotonicity we have $\Vdash^{F}_{S'} A[x/c]$ for every $S' \in F$. Regardless of the c we have picked, we will have $\Vdash^{F}_{S^c} A[x/c]$ for all $S^c \in F^c$, so by isomorphism we will have $\Vdash^{F^c}_{S^c} A[x/c]$ for all $S^c \in F^c$ and so $\Vdash^{F^c} A[x/c]$, contradicting the the assumption that $\nVdash^{F^c} A[x/c]$. Since this works for every choice of constant c, we obtain a contradiction, so we conclude $\nVdash_S \exists x A$ and then $\Vdash_S \exists x A$ implies $\Vdash_S A[x/a]$ for some $a \in N(S)$ by contraposition. □

As usual, we also have the following:

Theorem 3.5.13 The following equivalences hold:

1. $\Vdash_S \forall X_n A \iff \Vdash_{S'} A[X_n/P_n]$ for all $P_n \in N(S')$ in all $S \subseteq S'$, given $(n \geq 0)$;

2. $\Vdash_S \exists X_n A \iff \Vdash_S A[X_n/P_n]$ for some $P_n \in N(S)$, given $(n \geq 0)$.

Proof: Straightforward adaptation of the proof for Theorem 3.5.12, □

Since only properties of minimal logic were used, the results naturally extend also to classical and intuitionistic multibases.

With this we conclude that generalized S-validity of formulas is ultimately reducible to atomic derivability even if quantifiers are introduced in the language, so the semantics smoothly extends to predicate logic without losing its fundamental properties.

3.6 **Additional remarks**

3.6.1 Technical differences between multibases and models

The rich inner structure of bases is responsible for many of the differences between models and multibases. In Kripke semantics we need to specify domains, interpretations and accessibility relations, but in multibases all of this is done indirectly by bases themselves. As shown by our completeness proofs, Kripke models can be viewed as degenerate multibases with no interesting structure, composed only of axiomatic rules (to substitute assignments and interpretations) and vacuous rules (to specify domains and relations).

The structure of bases also enables use of the extension relation. Although use of a fixed relation instead of arbitrary ones may seem disadvantageous at first, we reap many benefits from its predetermined character. Preservation of atomic derivability would not hold in general were the relations to be arbitrary. This preservation also has as natural byproducts proof-theoretic equivalents of the monotonicity and domain preservation conditions, two properties that need to be externally imposed on Kripke models.

Generalized S-validity, perhaps the most interesting feature of multibase semantics, is so heavily reliant on base structure that it has no correspondent in Kripke semantics. It is also faithful to the original notion of S-validity as conceptualized by Prawitz and Dummett, proof of this being that almost all results expected to hold for S-validity hold for generalized S-validity. The new definitions also entirely avoid some problematic interactions between disjunction and implication, often responsible for the incompleteness results that have long plagued the literature.

It is also worthy of note that ours is a proof-theoretic semantics in more than one sense. First, the semantic evaluation of sentences is entirely determined by derivability in bases of a multibase, and generalized S-validity is entirely determined by derivability in bases – not even being bound by multibases. Second, everything done externally by functions in model-theoretic semantics is done internally by features of bases, so after bases have been defined everything else follows. Third, bases are clearly syntactic structures in the technical sense (see Section 1.4.2), even though multibases are clearly semantic in nature, which reinforces our point that semantics in terms of proofs are possible because the semantic nature of structures is given by universal quantification over their components, not the components themselves. Lastly, many of our results make clear use of proof-theoretic methods to show semantic results (the most notable being our proof of the Export principle in Theorems 3.3.19 and 3.5.10), which shows that the properties and structures studied in Proof Theory indeed have semantic bearing.

3.6.2 The philosophy of multibases

The philosophical principles underlying multibases are also entirely different from those justifying model-theoretic semantics, especially concerning the properties listed in Sections 1.2.2 and 1.2.3. The lack of inner structure in models is closely related to the strictly categorical nature of truth, as by introducing such components we would make evaluations conditional and thus introduce foreign counterfactual or epistemic elements. Problems created by the total character of truth are avoided in Kripke semantics by the introduction of modal elements capable of diversifying truth assignments, but its monolithic character is preserved by the flat nature of assignment functions. On the other hand, multibase semantics clearly has both categorical and hypothetical elements, as witnessed by the fact that, even though semantic clauses are defined categorically, consequence relations for generalized S-validity are entirely determined by hypothetical derivability (see Theorems 3.3.1 and 3.3.4). The non-total character of proof also makes it so that that semantic evaluations considering multiple bases and possible states of knowledge are entirely faithful to the initial concept, as opposed to *ad hoc* elements added to diversify a rigid concept of truth. Finally, the fragmentary nature of proofs is directly responsible for the concept of generalized S-validity; even though semantic values are determined by a single concept of proof, bases with different proofs of the same result may have very different semantic properties.

Some principles underlying multibases are also different from those of other proof-theoretic semantics. Unlike in Sandqvist's base-extension semantics (SANDQVIST, 2015), the semantic clauses are now defined categorically; the hypothetical nature of consequences is a product of the structure of bases, not of the clauses. We would still like to claim, however, that defining proofs categorically is not the same as defining them in terms of their introduction rules in natural deduction. The present author is entirely skeptical of the claim that only introduction rules are used in definitions of proof-theoretic validity. A more sensible claim would be that, in the definition ($\Vdash^M_S A \wedge B \iff \Vdash^M_S B$ and $\Vdash^M_S A$), the ($\Rightarrow$) direction is given by introduction rules, but ($\Leftarrow$) is given by elimination rules.

The arguments usually marshalled to justify the priority of introduction rules are undue extrapolations of an insight due to Gentzen, expressed in the following often cited passage (GENTZEN, 1969, pgs. 80-81):

> The introductions represent, as it were, the "definitions" of the symbols con-

> cerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only "in the sense afforded it by the introduction of that symbol". (...) By making these ideas more precise it should be possible to display the E-rules as unique functions of their corresponding I-rules, on the basis of certain requirements.

Introduction rules can only be seen as definitions of logical operators if we have previously established that they are to be paired with the strongest possible harmonic elimination rules. In this case, both introduction and elimination rules are unique functions of each other. Even so, this does not mean that elimination rules do not contribute to the meaning of operators, but only that our choice of introduction rules already determines which elimination rules should be chosen to complete the intended meaning. In other words, the semantic value of operators is given both by introduction and elimination rules, but if our interest lies in operators satisfying this condition the introduction rules may be taken as definitions of operators precisely because they also define their indispensable semantic complements. In view of this, multibase semantics differs from base-extension semantics inasmuch semantic clauses are defined categorically (as opposed to hypothetically), not in virtue of a supposed definition of semantic conditions exclusively in terms of introduction rules (as opposed to definitions making more explicit use of elimination rules, as is the case of Sandqvist's clause for disjunction).

Multibase semantics also departs from other similar approaches in its treatment of infinity. Even though both generalized S-validity and the original S-validity evaluate sentences in a base by considering its infinitely many possible extensions, we argue that generalized S-validity uses a *procedural* notion of infinity, as opposed to the *total* notion inadvertently invited by the original definition.

Although mathematically quite inelegant, our definition of focused multibases in terms of sequences (instead of sets) may help elucidate this claim. Focused multibases should be seen as representations of different points reached in an iterated process of knowledge extension. We have not included those characteristics in our definitions because this would unnecessarily complicate many proofs, but a focused multibase should ideally have no repetitions and no bases being preceded by its extensions. This would induce a tree-like structure in which at each point we are extending at least one of the tree's leafs, so at every step we are extending some previously obtained state of inferential knowledge. In this reading, focused multibases are to be viewed as pictures taken at particular moments in a dynamic process, not as static finished structures.

This can be represented graphically as follows:

Instant 1: $\langle S \rangle$

Instant 2: $\langle S, S' \rangle$

Instant 3: $\langle S, S', S'' \rangle$

(...)

The fact that we only need to consider finite multibases lends further credence to this reading. Since we, as finite beings, can only expand our states of knowledge finitely many times, each possible state of knowledge should be finite, even though infinitely many states are in principle reachable. We need to factor in all possible extensions in our evaluations, but *not at the same time*. A formula is then considered generally S-valid if it is valid *regardless of the state of knowledge we are currently in*. In contrast, the original notion of S-validity evaluates formulas in terms of a completed infinity containing all possible states of knowledge, so this iterated reading of extension procedures is lost.

CONCLUSION

The ideas presented in the early stages of proof-theoretic semantics were as elegant as they were bold. It seemed inconceivable that proofs, whose function was to lay grounds for the rigorous establishment of truths, would not only become independent entities, but also truth's equal. It is certainly unfortunate that many interesting proposals were shown to be lacking, but it's becoming increasingly clear that this was only a minor setback.

We hope it is not hubris to claim that multibase semantics is a worthy contribution to the field. Generalized S-validity does seem significantly close to S-validity, and it is certainly illuminating to see why it works and the original proposal doesn't. The power and flexibility of multibases becomes evident when it is shown to be a proof-theoretic alternative to Kripke models, perhaps the most widely used contemporary model-theoretic framework. The fact that proof-theoretic structures play a significant role on the inner workings of the semantics is also essential to its status as a proper proof-theoretic semantics – and I must confess that, as a proof theorist, I find it incredibly satisfying to prove semantic results using features of natural deduction.

This is hopefully only the beginning of a bigger project. Multibases are flexible enough to lend themselves to many purposes, and the fact that atomic restrictions can be used to materialize new concepts of proof leaves open the question of what other interesting semantics we may obtain. The framework does also seem to have considerable untapped potential, so perhaps there is still much to discover about the properties of multibases themselves. But this, of course, only the future will tell.

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