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Separability in Classical Natural Deduction

Rio de Janeiro
2025

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Dissertação apresentada, como requisito parcial para obtenção do título de Mestre, ao Programa de Pós-graduação em Filosofia, da Universidade do Estado do Rio de Janeiro. Área de concentração: Filosofia.

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INSCRIPTION

To my colleagues.

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"If anyone should think he has solved the problem of life & feels like telling himself that everything is quite easy now, he needs only to tell himself, in order to see that he is wrong, that there was a time when this "solution" had not been discovered; but it must have been possible to live then too & the solution which has now been discovered appears in relation to how things were then like an accident. And it is the same for us in logic too. If there were a "solution to the problems of logic (philosophy)" we should only have to caution ourselves that there was a time when they had not been solved (and then too it must have been possible to live and to think)"

Ludwig Wittgenstein. Culture and Value.
Blackwell Publishers Ltd, 1998.
Translated by Peter Winch

ABSTRACT

WADDINGTON, J.T. *Separability in Classical Natural Deduction*. 2025. 85 f. Dissertação (Mestrado em Filosofia) – Instituto de Filosofia e Ciências Humanas, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, 2025.

This dissertation investigates the separation property in natural deduction systems for propositional classical logic. The separation property states that given a system S, S has the separation property if, whenever Π is a Normal Deduction of A from Γ in S, then the only inference rules that are applied in Π are the inference rules for the logical constants that occur in A or in some formula of Γ. The separation property is closely related to the subformula property. In fact, the separation property is a corollary of the subformula property, but the converse is not the case. There are systems for propositional classical logic in natural deduction that have the separation property but don't have the subformula property. This dissertation instigates a discussion about grades of analyticity, where the subformula property engenders a stronger sense of analyticity than the separation property. The first chapter of the dissertation focuses on the history of natural deduction systems, from Gentzen and Jaśkowski to Prawitz. The second chapter focuses on the separation property and contains (I) a history of the separability theorem for intuitionistic logic, (II) a discussion about separability in natural deduction systems for propositional classical logic and, (III) a discussion about different grades of analyticity. The third chapter focuses on separable systems for classical propositional logic that are obtained from an intuitionistic system by the addition of structural rules. The fourth chapter focuses on the development of a separable natural deduction system for classical propositional logic that is obtained through the addition of an implicational rule to an intuitionistic natural deduction system. The NH system is obtained through the addition of Hosoi's rule $((A \to B) \to B), (A \to C), (B \to C) \vdash C$ to the propositional fragment of Gentzen's NJ system. The main results of this dissertation where (I) The development of the NH system, (II) the development of a normalization procedure for the NH system, and (III) the classification of different systems into three grades of analyticity: (i) non analytical systems (systems that have neither the subformula nor the separation property), (ii) strictly analytical systems (systems that have the separation property but don't have the subformula property) and (iii) Ultra Strictly Analytical systems (systems that have the separation property and the subformula property).

Keywords: natural deduction; intuitionistic logic; classical logic; separability.

RESUMO

WADDINGTON, J.T. *Separabilidade para Dedução Natural Clássica*. 2025. 85 f. Dissertação (Mestrado em Filosofia) – Instituto de Filosofia e Ciências Humanas, Universidade do Estado do Rio de Janeiro, Rio de Janeiro, 2025

Esta dissertação investiga a propriedade de separabilidade em sistemas de dedução natural para a lógica proposicional clássica. A propriedade da separabilidade afirma que, dado um sistema S, S é separável se, sempre que Π é uma Dedução Normal de A a partir de Γ em S, então as únicas regras de inferência que são aplicadas em Π são as regras de inferência para as constantes lógicas que ocorrem em A ou em alguma fórmula de Γ. A propriedade de separabilidade está intimamente relacionada à propriedade do subfórmula. De fato, a separabilidade é um corolário da propriedade do subfórmula, mas o inverso não é verdade. Existem sistemas para a lógica proposicional clássica em dedução natural que são separáveis, mas não têm a propriedade do subfórmula. Isso instiga uma discussão sobre graus de analiticidade, onde a propriedade do subfórmula gera um sentido mais forte de analiticidade do que a propriedade de separação. O primeiro capítulo da dissertação foca na história dos sistemas de dedução natural, de Gentzen e Jaśkowski a Prawitz. O segundo capítulo foca na propriedade de separabilidade e contém (I) uma história do teorema de separabilidade para a lógica intuicionista, (II) uma discussão sobre separabilidade em sistemas de dedução natural para a lógica proposicional clássica e, (III) uma discussão sobre diferentes graus de analiticidade. O terceiro capítulo foca em sistemas separáveis para a lógica proposicional clássica que são obtidos a partir de um sistema intuicionista pela adição de regras estruturais. O quarto capítulo foca no desenvolvimento de um sistema de dedução natural separável para a lógica proposicional clássica que é obtido através da adição de uma regra implicacional a um sistema de dedução natural intuicionista. O sistema NH é obtido através da adição da regra do Hosoi $((A \to B) \to B), (A \to C), (B \to C) \vdash C$ ao fragmento proposicional do sistema NJ de Gentzen. Os principais resultados desta dissertação foram (I) o desenvolvimento do sistema NH, (II) o desenvolvimento de um procedimento de normalização para o sistema NH, e (III) a classificação de diferentes sistemas em três graus de analiticidade: (i) sistemas não analíticos (sistemas que não têm nem a propriedade do subfórmula nem são separáveis), (ii) Sistemas Estritamente Analíticos (sistemas que são separáveis, mas não têm a propriedade do subfórmula) e (iii) sistemas Ultra Estritamente Analíticos (sistemas que são separáveis e têm a propriedade do subfórmula).

Palavras-chave: dedução natural; lógica intuicionista; logica clássica; separabilidade.

CONTENTS

1 NATURAL DEDUCTION SYSTEMS

1.1 Gentzen and Jaśkowski

In 1934, two papers were published: "On the Rules of Suppositions in Formal Logic" and "Untersuchungen uber das logischen Schliessen." These two papers announced what we now recognize as "Natural Deduction". The authors were, respectively, Stanisław Jaśkowski, a Polish logician from Warsaw, and Gehrard Gentzen, a German logician from Göttingen. These "two authors had never been in contact with one another and they had (apparently) no common intellectual background that would otherwise account for their mutual interest in this topic" [1]. The term "Natural Deduction" originally, "das natürliche Schließen" was coined by Gentzen. Jaśkowski called it "method of suppositions" [2].

Jaśkowski's paper begins as follows:

> *In 1926, Prof. J. Łukasiewicz called attention to the fact that mathematicians in their proofs do not appeal to the theses of the theory of deduction, but make use of other methods of reasoning. The chief means employed in their method ist that of an arbitrary supposition. The problem raised by Mr. Łukasuewicz was to put these methods unter the form of deduction. The present paper contains the solution of that problem.*(JAśKOWSKI, 1934, §1)

The challenge proposed by Łukasiewicz was to construct a system that better reflected mathematical practice, particularly the use of arbitrary assumptions. Axiomatic Systems, such as the one developed by Frege [1879] in the "Begriffsschrift" or Łukasiewicz P_2 system, were far from the human mathematical practice. A common example is the derivation of the tautology $p \to p$ in P_2:

Axioms

- $p \to (q \to p)$ (I)
- $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ (II)
- $(\neg p \to \neg q) \to (q \to p)$ (III)

Proof

1. $p \to ((q \to p) \to p)$ (instance of I)

[1] see (PEREIRA et al., 2023, pg.1) and Pelletier [(PELLETIER, 1999, pg.1)
[2] see Pelletier (PELLETIER; HAZEN, 2012)pg.347

2. $(p \to ((q \to p) \to p)) \to ((p \to (q \to p)) \to (p \to p))$ (instance of II)

3. $(p \to (q \to p)) \to (p \to p)$ (modus ponens 1. and 2.)

4. $p \to (q \to p)$ (instance of II)

5. $p \to p$ (modus ponens 3. and 4.)

It takes five lines of reasoning to deduce $p \to p$ and this proof doesn't start from assumptions. In this type of system, axioms are the building blocks of proofs.

In Jaśkowski "method of supposition", one is allowed to make assumptions, marked by the letter S [3]. There are two rules that allows one to "discharge" assumptions: the conditional proof or conditional introduction and the *Reductio ad Absurdum* or indirect proof. Besides the rules that discharge hypothesis, he had rules for the manipulation of formulas, such as modus ponens[4]. Suppositions are the building blocks of proofs! Pelletier enumerates the following remarks from Jaśkowski about his "method of supposition":

- Jaśkowski mentions (p. 238) that the system "has the peculiarity of requiring no axioms" (PELLETIER, 1999, pg.4)
- his system is "more suited to the purposes of formalizing practical [mathematical] proofs" than were the then-accepted system, which are "so burdensome that [they are] avoided even by the authors of logical [axiomatic] systems." (PELLETIER, 1999, pg.4)
- "in even more complicated theories the use of [the axiomatic method] would be completely unproductive." (PELLETIER, 1999, pg.4)

Pelletier's conclusion is that: *Given all this, one could say that Jaśkowski was the inventor of natural deduction as a complete logical theory.* (PELLETIER, 1999, pg.4)

Independently, Gentzen developed his own system of natural deduction. His opening remarks in the 1934/35 paper [5], "Untersuchungen über das logische Schließen", as referenced by (PELLETIER; HAZEN, 2012) and (PEREIRA et al., 2023) are:

> My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return.

[3] Jaśkowski uses polish notation. It is a prefixed language where, C stands for $\to$ and N stands for $\neg$
[4] see (PELLETIER; HAZEN, 2012, pg.384)
[5] Original in (GENTZEN, 1935). For english translation see (GENTZEN, 1964)

> In contrast, I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a 'calculus of natural deduction' ('NJ' for intuitionist, 'NK' for classical predicate logic). ... (GENTZEN, 1935)[6]

As noted by Pereira (PEREIRA et al., 2023, pg.3), Gentzen also states the same motivation at the beginning of Section II:

> We wish to set up a formalism that reflects the logical reasoning involved in mathematical proofs as accurately as possible.(GENTZEN, 1935)[7]

Gentzen had a similar goal as Jaśkowski: formulate a system that allows one to make assumptions, not just categorical assertions:

> ...the essential difference between NJ-derivations and derivations in the systems of Russell, Hilbert, and Heyting is the following: In the latter systems true formulae are derived from a sequence of 'basic logical formulae' by means of a few forms of inference. Natural deduction, however, does not, in general, start from basic logical propositions but rather from assumptions to which logical deductions are applied. By means of a later inference the result is then again made independent of the assumption. [Gentzen ibid Pelletier (PELLETIER; HAZEN, 2012) pg.348]

But, Gentzen had a further motivation: his aim was not just to create a system that "reflects the logical reasoning". As noted in (PEREIRA et al., 2023, pg.3) "this starting point was just a preparatory step to a more important investigation":

> A closer investigation of the specific properties of the natural calculus finally led me to a very general result which will be referred to below as the 'Hauptsatz'.(GENTZEN, 1935)[8]

The "Hauptsatz," also known as the "Cut elimination theorem," is significant because of its corollary, the "subformula property" [9]. This property says something about the analyticity of proofs [10]:

> No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of that result.(GENTZEN, 1935)[11]

[6] English translation in (GENTZEN, 1964, pg.288)
[7] English translation in (GENTZEN, 1964, pg.291)
[8] English translation in (GENTZEN, 1964, pg.289)
[9] Professor Elaine Pimentel pointed out that this holds only for Gentzen's original system. There are systems satisfying the Hauptsatz that don't satisfy the subformula property
[10] see (PEREIRA et al., 2023, pg.3)
[11] English translation in (GENTZEN, 1964, pg.289)

The subformula property guarantees that logical proofs are analytic in the sense that every formula used in the derivation of a theorem is a subformula of that theorem. This property shows that every derivation Π of A from Γ can be transformed into a derivation Π^* of A from Γ where only subformulas of A or of some formula present in Γ are used. This, in a certain sense, means that every information used in order to prove something is already contained in the final conclusion of the proof: they are subformulas of the conclusion. As Gentzen puts it:

> "The Hauptsatz says that every purely logical proof can be reduced to a determinate, though not unique, normal form. Perhaps we may express the essential properties of such a normal proof by saying "it is not roundabout." No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of that result." (GENTZEN, 1935) [12]

The Hauptsatz, announced Gentzen, couldn't be proved using the natural calculus. It was put aside, considered "unsuitable". This led Gentzen to develop another system, the sequent calculus. With this new conceptual apparatus, he could prove the Hauptsatz for intuitionistic and classical predicate logic:

> "In order to be able to enunciate and prove the Hauptsatz in a convenient form, I had to provide a logical calculus especially suited to the purpose. For this the natural calculus proved unsuitable. For, although it already contains the properties essential to the validity of the Hauptsatz, it does so only with respect to its intuitionist form, in view of the fact that the law of excluded middle, as pointed out earlier, occupies a special position in relation to these properties." (GENTZEN, 1935) [13]

In 2005 Von Plato found an early handwritten version of Gentzen's thesis (VON.PLATO; GENTZEN, 2008):

> "We found in February 2005 an early handwritten version of Gentzen's thesis, with exactly the above title, but with rather different contents: Most remarkably, it contains a detailed proof of normalization for what became the standard system of natural deduction. The manuscript is located in the Paul Bernays collection at the ETH-Zurich with the signum Hs. 974:271. Bernays must have gotten it well before the time of his being expelled from Gottingen on the basis of the racial laws in April 1933. He seems to have never mentioned the existence of a proof of normalization for natural deduction by Gentzen, even if he discussed Gentzen's work extensively [Bernays 1965, 1970] and was also fully aware of the published proof of normalization by Prawitz [1965]."

[12] English translation in (GENTZEN, 1964, pg.298)
[13] English translation in (GENTZEN, 1964, pg.289)

As Pereira points out (PEREIRA et al., 2023, pg.16), this version had quite a different content from the published version of the thesis:

> Gentzen's original plan for the thesis seems to have been the following: First set up the calculus of natural deduction, a rule of induction included (chapter I). Next show that the calculus is equivalent to an axiomatic calculus (II). Then show normalization for intuitionistic natural deduction (III). In the next step, show that classical arithmetic, and especially the question of its consistency, reduces to intuitionistic arithmetic (IV). Finally, extend the subformula property of natural deduction to the system of intuitionistic arithmetic, to prove the consistency of the latter (V). (VON.PLATO; GENTZEN, 2008, pg.241)

Gentzen had a proof of the normalization theorem for intuitionistic predicate logic, but because of the problems faced with classical case, he abstained from publishing it. It was only thirty years later, in 1965, that Prawitz independently obtained and published a proof of normalization [14] for classical and intuitionistic logic[15] . In the same year, 1965, Raggio also obtained a normalization proof for NK. [16]

1.1.1 Gentzen's NJ and NK

1.1.1.1 NJ

When Gentzen announces his calculus of natural deduction in Section II of the "Untersuchungen Über das Logischen Schliessen", he first presents a system restricted to intuitionistic reasoning. He calls it NJ:

> 2.1 We intend now to present a calculus for "natural" intuitionistic derivations of valid formulae. The restriction to intuitionistic reasoning is only provisional; we shall explain below (cf. §5) our reasons for doing so and shall show in what way the calculus has to be extended for classical reasoning (by including the law of the excluded middle)

Gentzen's NJ system consists of two types of rules: introduction rules ($I - rules$) and elimination rules ($E - rules$). The ExFalso rule can be seen as the rule that eliminates $\perp$. $\perp$ doesn't have an introduction rule. [17]

[14] Among other important results obtained by Prawitz
[15] see Peletier (PELLETIER; HAZEN, 2012) pg.371
[16] see (RAGGIO, 1965)
[17] One way to understand it is through the fact that there is no proof of the $\perp$

> To every logical symbol $\rightarrow, \&, \vee, \forall, \exists, \neg$, belongs precisely one inference figure which ”introduces the symbol - and one which eliminates it.”

The rules are:

- Conjunction [18]

$$I - \wedge \frac{A \qquad B}{A \wedge B} \qquad\qquad E - \wedge r \frac{A \wedge B}{B} \qquad\qquad E - \wedge l \frac{A \wedge B}{A}$$

- disjunction

$$I - \vee r \frac{A}{A \vee B} \qquad I - \vee l \frac{B}{A \vee B} \qquad E - \vee l \frac{A \vee B \quad \begin{matrix}[A]^1 \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B]^2 \\ \vdots \\ C\end{matrix}}{C} 1, 2$$

- Implication

$$I\!-\!\rightarrow \frac{\begin{matrix}[A]^1 \\ \vdots \\ B\end{matrix}}{A \rightarrow B} 1 \qquad\qquad E\!-\!\rightarrow \frac{A \qquad A \rightarrow B}{B}$$

- negation

$$I - \neg \frac{\begin{matrix}[A]^1 \\ \vdots \\ \bot\end{matrix}}{\neg A} 1 \qquad\qquad E - \neg \frac{A \qquad \neg A}{\bot}$$

- $\forall$ [19]

$$I - \forall \frac{Fa}{\forall x F x} \qquad\qquad E - \forall \frac{\forall x F x}{Fa}$$

- $\exists$ [20]

$$I - \exists \frac{Fa}{\exists x F x} \qquad\qquad E - \exists \frac{\exists x F x \qquad \begin{matrix}[Fa]^1 \\ \vdots \\ C\end{matrix}}{C} 1$$

---

[18] Gentzen uses $\&$ instead of $\wedge$ as a symbol for conjunction

[19] Restriction on the introduction rule: the variable a must not occur in the formula represented in the schema by $\forall x F x$, nor in any assumption formula upon which that formula depends

[20] Restriction on the elimination rule: the variable a must not occur in the formula represented in the schema by $\exists x F x$; nor in an upper formula represented by C; nor in any assumption formula upon which that formula depends, with the exception of the assumption formulae represented by Fa correlated with the $E - \exists$

- *ExFalso* [21]

$$\frac{\perp}{A}$$

1.1.1.2 I-rules and E-rules

§5 of section II of the "Untersuchen" opens with a series of comments about NJ. Gentzen states that:

> 5.1 The calculus NJ lacks a certain formal elegance. This has to be put against the following advantages :
> 5.11. A close affinity to actual reasoning, which had been our fundamental aim in setting up the calculus. The calculus lends itself in particular to the formalization of mathematical proofs.
> 5.12. In most cases the derivations for valid formulae in our calculus are shorter than their counterparts in logistic calculi [axiomatic systems]. This is so primarily because in logistic derivations one and the same formula usually occurs a number of times (as part of other formulae), whereas this happens only very rarely in the case of NJ-derivations.

And, most importantly, it is stated that:

> 5.13 "The introduction represents, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more in the final analysis than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'(Gentzen34/35)[22]".

This passage is of the uttermost relevance. It presents the idea that rules, and not truth

[21] If the Ex falso rule is omitted, the resulting system is a system for minimal logic
[22] english translation in (GENTZEN, 1964, pg.295)

conditions, are, in some sense, “definitions” of the logical constants. [23] Pereira stresses that *“In addition to the use of assumptions, a second fundamental (and perhaps more critical for Gentzen’s purposes) feature of Gentzen’s natural deduction calculi is that the deductive role of the logical operators is determined in these calculi not by axioms but by a system of rules”*.[24].

1.1.1.3 NK

To obtain a “complete classical calculus”, Gentzen extends NJ by the addition of *Tertium non Datur*. [25]:

> From the calculus NJ we obtain a complete classical calculus NK by adding the ”law of excluded middle” (tertium non datur), i.e: As initial formulae of a derivation we now also allow in addition to the assumption formulae “basic formulae” of the form $A \vee \neg A$, where A is said to be replaced by an arbitrary formula. (Gentzen 1934/35)[26]

Strangely enough, Gentzen chooses a “basic formula”, not a rule, to obtain classical logic. His justification is that this formulation seems the “most natural”[27]. He considers introducing a new inference schema:

$$\frac{\neg\neg A}{A}\, DNE$$

But, he argues that the double negation elimination rule does not follow directly from his method of introducing negation. The rule of double negation elimination would occupy a strange place in his system: negation would have two elimination rules, where one of them does not ”follow at all” from the introduction rule of negation:

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[23] The expression “in some sense” is used here because of Prof.Pereira’s remark in (PEREIRA et al., 2023):

> Gentzen does not claim that the introductions are definitions, but that they represent, as it were, “definitions”, and the word “definitions” being placed in quotation marks could mean that Gentzen would not commit himself to a definition in a strict sense. Be that as it may, we cannot deny that we have here the seed of the following idea: the meaning of a logical operator is determined/fixed by its introduction rules, and the elimination rules would be a consequence of this meaning fixed by the introduction rules. The introduction rules did not need justification, as they would be meaning constitutive; the elimination rules would be justified by the introduction rules.

[24] (PEREIRA et al., 2023, pg.8)

[25] it is worth noting that in natural deduction classical logic is ”built upon” intuitionistic logic. This isn’t always the case: Tableaux for classical logic seems simpler than for intuitionistic logic

[26] english translation in (GENTZEN, 1964, pg.295)

[27] (GENTZEN, 1964, pg.295)

> *"However, such a schema [DNE] still falls outside of the framework of the NJ-inference figures, because it represents a new elimination of the negation whose admissibility does not follow at all from our method of introducing the $\neg$ symbol by the $\neg - I$"* [28].

1.1.2 Prawitz's 1965 "Natural Deduction"

Prawitz's 1965 doctoral dissertation, "Natural Deduction" (PRAWITZ, 1965), is the next seminal work in the history of natural deduction. The preface to the Dover edition reads as follows:

> "The main theme of the monograph is the establishing of a certain normal form for derivations in the systems of natural deduction that were introduced by Gentzen in his doctoral dissertation "Untersuchungen über das logische Schliessen" in 1934. The result, that every derivation in a system of natural deduction can be transformed to this normal form by certain reductions defined in the monograph, is known as the *normalization theorem* for the system in question, a term commonly used since the time of the above-mentioned Proceedings (following a suggestion by Georg Kreisel). It is equivalent to what is known as the *Hauptsatz for Gentzen's corresponding calculi of sequents*" (PRAWITZ, 1965, pg.V)

In his thesis, Prawitz obtained, among other important results, the "*normalization theorem"* for intuitionistic [29] and classical logic. This result guarantees that every proof can be transformed into a "normal form". The procedure eliminates detours, i.e, unnecesary information that is used in the proof.

In NJ [30] there, apparently [31], are two types of Detour: *Maximum Formulas* and *Maximum Segments*:

- A formula occurence in a deduction Π that is the consequence of an application of an I- or $\perp$- rule and a major premiss of an application of an E-rule is said to be a *maximum formula* in Π. (PRAWITZ, 1965, pg.34)

[28] (GENTZEN, 1964) pg.295

[29] As it was stated, Gentzen obtained a normalization result for intuitionistic logic, but didn't publish it. The result was made public by Von Plato (VON.PLATO; GENTZEN, 2008) in 2005

[30] Also works for NM

[31] "Apparently" here is being used because *maximum* formulas are actually special cases of *maximum segments*

- Maximum segment is a segment[32] that begins with a consequence of an application of an I-rule or the I-$\perp$ rule and ends with a major premiss of an E-rule. Note that a maximum formula as defined in Chapter III is a special case of a maximum segment" (PRAWITZ, 1965, pg.49)

As we will see, Natural Deduction Systems for classical logic may present new types of detour. The addition of a classicizing rule to NJ isn't free of complications. In such cases, additional reduction procedures are needed. In this section a brief explanation of Prawitz's results for NJ and C[33] will be presented.

1.1.2.1 Prawitz's results for intuitionistic logic

In section II of "Natural Deduction" Prawitz presents a procedure[34] to *"remove a formula occurence that is the consequence of an I-rule and the major premiss of an E-rule."* [35], i.e, a procedure that removes *maximal formulas.* [36]:

- $\wedge$ reduction

$$\dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ A\end{matrix} \qquad \begin{matrix}\Sigma_2 \\ B\end{matrix}}{A \wedge B}}{\begin{matrix}A \\ \Sigma_3\end{matrix}} \qquad \Longrightarrow \qquad \begin{matrix}\Sigma_1 \\ A \\ \Sigma_3\end{matrix}$$

- $\vee$ reduction [37]

[32] Prawitz defines a segment in pg.49 of (PRAWITZ, 1965):
A segment in a deduction Π *is a sequence* $A_1, A_2, ..., A_n$ *of consecutive formula occurences in a thread in* Π *such that (i)* A_1 *is not the consequence of an application of* $\vee$*E of* $\exists$*E, (ii)* A_i*, for each* $i < n$*, is a minor premiss of an application of* $\vee$*E or* $\exists$*E ; and (iii)* A_n *is not the minor premiss of an application of* $\vee$*E or* $\exists$*E*

[33] Prawitz's classical system is different from Gentzen's NK. He chooses the *reductio ad absurdum* rule:

$$\dfrac{\begin{matrix}[\neg A]^1 \\ \Pi \\ \perp\end{matrix}}{A}\ 1$$

instead of the basic formula "$A \vee \neg A$"

[34] see (PRAWITZ, 1965) pg.36-37

[35] (PRAWITZ, 1965) pg.35

[36] see (PRAWITZ, 1971, pg.252-255)

[37] "Here $[Ai]$ denotes the set of assumptions in Σ_i that are closed by the $\vee E$ in question (i = 1 or 2)." (PRAWITZ, 1971)pg.252

$$\dfrac{\dfrac{\begin{matrix}\Sigma\\ A_i\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}[A_1]\\ \Sigma_1\\ B\end{matrix} \qquad \begin{matrix}[A_2]\\ \Sigma_2\\ B\end{matrix}}{B} \qquad \Longrightarrow \qquad \begin{matrix}\Sigma\\ [A_i]\\ \Sigma_i\\ B\end{matrix}$$

- $\rightarrow$ reduction [38]

$$\begin{matrix}\dfrac{\begin{matrix}\Sigma_1\\ A\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \Sigma_2\\ B\end{matrix}}{A \rightarrow B}}{B}\\ \Sigma_3\end{matrix} \qquad \Longrightarrow \qquad \begin{matrix}\Sigma_1\\ [A]\\ \Sigma_2\\ B\\ \Sigma_3\end{matrix}$$

- $\forall$ reduction [39]

$$\begin{matrix}\Sigma_1\\ \dfrac{\dfrac{A}{\forall x A^a_x}}{A^{ax}_{xt}}\\ \Sigma_2\end{matrix} \qquad \Longrightarrow \qquad \begin{matrix}\Sigma^a_{1t}\\ A^a_t\\ \Sigma_2\end{matrix}$$

- $\exists$ reduction [40]

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\Sigma_1\\ A(^x_t)\end{matrix}}{\exists x A} \qquad \begin{matrix}[A(^x_a)]\\ \Sigma_2\\ B\end{matrix}}{(B)}\\ \Sigma_3\end{matrix} \qquad \Longrightarrow \qquad \begin{matrix}\Sigma_1\\ [A^x_t]\\ {\Sigma_2}^a_t\\ (B)\\ \Sigma_3\end{matrix}$$

The five reductions above correspond to the five possible forms of maximum formula, as stated in (PRAWITZ, 1971), pg. 251. But, to be able to also remove maximal segments, Prawitz introduces two permutative reductions. The aim of these reductions is, as the name suggests, to permute the application of the rule in question with other rules [41]:

- $E\vee$ permutation

[38] "Here [A] denotes the set of assumptions in Σ_2 that are closed by the $\rightarrow -I$."(PRAWITZ, 1971)pg.252.

[39] *"We write $\Sigma(a)$ to indicate that the formulas in the part of the derivation above $A(a)$ may contain the parameter a. The deduction $\Sigma(t)$is to be obtained from $\Sigma(a)$ by replacing every occurrence of a by the term t. Note that the restriction on $\forall - I$ and the tacit assumption about the proper parameters (sec. 1.2.4) together guarantee that the right derivation is correct."(PRAWITZ, 1971)pg.252.*

[40] *"The remark made above in connection with the $\vee$-reduction applies also here mutatis mutandis. $[A(a)]$ denotes the set of assumptions in $\Sigma_2(a)$ which are closed by the $\exists E$."(PRAWITZ, 1971)pg.252.*

[41] The permutation procedures are presented in section IV of (PRAWITZ, 1965, pg.51)

$$\dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ A \vee B\end{matrix} \quad \begin{matrix}\Sigma_2 \\ C\end{matrix} \quad \begin{matrix}\Sigma_3 \\ C\end{matrix}}{C} \qquad \Sigma_4}{D} \quad \Longrightarrow \quad \dfrac{\begin{matrix}\Sigma_1 \\ A \vee B\end{matrix} \quad \dfrac{\begin{matrix}\Sigma_2 \\ C\end{matrix} \quad \Sigma_4}{D} \quad \dfrac{\begin{matrix}\Sigma_3 \\ C\end{matrix} \quad \Sigma_4}{D}}{D}$$

- $E\exists$ permutation

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\Sigma_1 \\ \exists x B\end{matrix} \quad \begin{matrix}\Sigma_2 \\ C\end{matrix}}{C} \qquad \Sigma_4}{(D)} \\ \Sigma_5\end{matrix} \quad \Longrightarrow \quad \begin{matrix}\dfrac{\begin{matrix}\Sigma_1 \\ \exists x B\end{matrix} \quad \dfrac{\begin{matrix}\Sigma_2 \\ C\end{matrix} \quad \Sigma_4}{D}}{(D)} \\ \Sigma_5\end{matrix}$$

Using these reduction procedures Prawitz proves the following theorems concerning the normal form of derivations in the NJ system:

> ***Theorem I.*** *If* $\Gamma \vdash A$ *holds in the system for intuitionistic or minimal logic, then there is a normal deduction in this system of A from* Γ (PRAWITZ, 1965, pg.50)

The above theorem entails two corollaries that are central to this work:

> ***Corollary I.*** *(subformula principle) Every formula occuring in a normal deduction in I or in M of A from* Γ *is a subformula of A or of some formula of* Γ. (PRAWITZ, 1965, pg.53)
>
> ***Corollary IV.*** *(separation theorem) If* Π *is a normal deduction in I or in M of A from* Γ*, then the only inference rules that are applied in* Π *are the inference rules in I or in M for the logical constants that occur in A or in some formula of* Γ (PRAWITZ, 1965, pg.54)

The natural deduction system for intuitionistic logic has both (I) the subformula property and (II) the separation theorem. [42]

1.1.2.2 Prawitz's results for classical logic

Prawitz's system C is obtained through the addition of the following rule to NJ:

[42] The subformula property entails the separation theorem, but the separation theorem doesn't entail the subformula, see the footnote on (PRAWITZ, 1965) pg.54 . This dissertation focuses on classical systems that have the separation property, but not the subformula property. The NH system, presented in section 5. is an example of a separable system that doesn't have the subformula principle.

- $\perp_c$

$$\dfrac{\begin{array}{c}[\neg A]^1\\ \vdots\\ \perp\end{array}}{A}\,1$$

The addition of $\perp_c$ to NJ may result in a new type of detour:

Definition 1.1.1 ***Classical Detour.*** *The conclusion of an application of $\perp_c$ is the major premise of an elimination rule.*

Prawitz's strategy is the following: reduce every application of classical reasoning to atomic sentences. To obtain this result, Prawitz restricts his language to $<\rightarrow, \wedge, \neg, \forall, \perp>$. $\vee$ and $\exists$ are left out:

Consider a derivation Π of the following form:

$$\dfrac{\begin{array}{c}[\neg F]\\ \Sigma\\ \perp\end{array}}{F}$$

There are four subcases to examine. F can be $\neg A$, $A \wedge B$, $A \rightarrow B$ or $\forall x B$. The following procedures reduce, at each application, the degree of the formulas on which $\perp_c$ is applied:

- $\wedge$

$$1\,\dfrac{\begin{array}{c}[\neg(B\wedge C)]^1\\ \Sigma\\ \perp\end{array}}{(B\wedge C)} \quad\Longrightarrow\quad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[B\wedge C]^1}{B}\quad [\neg B]^2}{\perp}}{[\neg(B\wedge C)]}\,1\\ \Sigma\\ \perp\end{array}}{B}\,2 \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[B\wedge C]^3}{C}\quad [\neg C]^4}{\perp}}{[\neg(B\wedge C)]}\,3\\ \Sigma\\ \perp\end{array}}{C}\,4}{B\wedge C}$$

- $\rightarrow$

$$\dfrac{\begin{array}{c}[\neg(B\rightarrow C)]\\ \Sigma\\ \perp\end{array}}{(B\rightarrow C)} \quad\Longrightarrow\quad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[B]^1\quad [B\rightarrow C]^2}{C}\quad [\neg C]^3}{\perp}}{[\neg(B\rightarrow C)]}\,2\\ \Sigma\\ \perp\end{array}}{C}\,3}{(B\rightarrow C)}\,1$$

- $\forall$

$$\dfrac{\begin{array}{c}[\neg\forall xB]\\ \Sigma\\ \bot\end{array}}{\forall xB} \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[\forall xB]^1}{B^x_b} \quad [\neg B^x_b]^2}{\bot}}{[\neg\forall xB]}\,1\\ \Sigma\\ \bot\end{array}}{B^x_b}\,2}{(\forall xB)}$$

Where b is to be a parameter that does not occur in Π

- $\neg$

There is no need for a $\neg$ reduction procedure. There is no classical detour. If A has the form of $B \rightarrow \bot$, the application of the $\bot_c$ rule can be replaced by an application of the $I \rightarrow$ rule, an intuitionistic rule. This is explained in (PRAWITZ, 1965, pg.20) :

> "*restriction on the $\bot_c$-rule* : A is not to have the form of $B \rightarrow \bot$. This restriction is also only a matter of convenience. That nothing is lost by this restriction can easily be seen. Suppose we have an application of the $\bot_c$-rule that does not satisfy the restriction. Then the assumption $(B \rightarrow \bot) \rightarrow \bot$ discharged by this application can be replaced with the following deduction of $(B \rightarrow \bot) \rightarrow \bot$ from B:
>
> $$\dfrac{\dfrac{B \qquad [B \rightarrow \bot]^1}{\bot}}{(B \rightarrow \bot) \rightarrow \bot}\,1$$
>
> By this replacement, the given application of the $\bot_c$-rule is turned into an application of the $\rightarrow$-I rule discharging the assumption B." (Prawitz 1965 (PRAWITZ, 1965)pg.20)

It can be seen from the reduction procedures above that the subformula principle cannot be fully satisfied. This phenomena happens because of two reasons: (i) in Prawitz's system, classical reasoning is governed by the behavior of negation:

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots\\ \bot\end{array}}{A}$$

and (ii), even if classical reasoning is restricted to atomic applications, the negation of these atoms aren't always subformulas of the final conclusion. Take the canonical proof of Peirce's formula as an example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^2 \qquad [\neg p]^1}{\bot}}{q}}{p \to q}\,2 \qquad [(p \to q) \to p]^3}{p} \qquad [\neg p]^1}{\dfrac{\bot}{p}\,1}}{((p \to q) \to p) \to p}\,3$$

Classical reasoning is only being applied to the atom p. But, the formula $\neg p$ isn't a subformula of the conclusion $(((p \to q) \to p) \to p)$. Because of the reasons discussed above, the "Subformula principle" obtained for classical logic in chapter III isn't quite the same as the one obtained for intuitionistic logic. There is an exception:

> "Corollary I. (subformula principle.) *Every formula occurrence in a normal deduction in C' of A from* Γ *has the shape of a subformula of A or of some formula of* Γ*,* ***except*** *for assumptions discharged by applications of the* $\bot_c$ *rule and for occurrences of* $\bot$ *that stand immediately below such assumptions* (Prawitz 1965 (PRAWITZ, 1965) pg.42)[43]"

Prawitz's classical system also breaks the separability property: the weakened version of the subformula principle doesn't entail separability, Peirce's axiom cannot be proved using only implicational rules. In Prawitz's system C, one must use a rule of negation to obtain the complete classical fragment of $[\to]$. This leads us to the main theme of this dissertation: **Can there be a formulation of propositional classical logic in natural deduction that is separable?**

[43] Bold letters not in the original

2 SEPARABILITY

The separation property (or separability) states that, given any theorem A of a calculus Θ, there is a proof of A in Θ where only rules (or axioms) involving the logical connective(s) present in A are used. But why is this desirable?

If one follows Dummett and Gentzen in the idea that rules determine the meaning of the logical constants:

> ”The meaning of a mathematical statement determines and is exhaustively determined by its use. The meaning of such a statement cannot be, or contain as an ingredient, anything which is not manifest in the use made of it, lying solely in the mind of the individual who apprehends that meaning: if two individuals agree completely about the use to be made of the statement, then they agree about its meaning. The reason is that the meaning of a statement consists solely in its role as an instrument of communication between individuals, just as the powers of a chesspiece consist solely in its role in the game according to the rules.” (DUMMETT, 1975, pg.6)

> ”5.13 ”The introduction represents, as it were, the ‘definitions’ of the symbols concerned, and the eliminations are no more in the final analysis than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only ‘in the sense afforded it by the introduction of that symbol’.(GENTZEN, 1935) [44]

One could argue that the separation property guarantees that the meaning of logical constants is independent from each other. As Tennant puts it:

> ”...the analytic project must take the operators one-by-one. The basic rules that determine logical competence must specify the unique contribution that each operator can make to the meanings of complex sentences in which it occurs, and derivatively, to the validity of arguments in which such sentences occur. This is the requirement of separability.
> It follows from separability that one would be able to master various fragments of the language in isolation, or one at a time. It should not matter in what order one learns... the logical operators. It should not matter if indeed some operators are not yet within one’s grasp. All that matters is that one’s grasp of any operator should be total simply on the basis of schematic rules governing instances involving it.”
> (TENNANT, 1997, pg.315) ibid. (SHAPIRO, 1998, pg.601)

[44] english translation in (GENTZEN, 1964, pg.295))”

If one accepts that the basic rules of inference **determine** the meaning of the logical constants, than the separation property guarantees that the meaning of each logical operator doesn't depend on the meaning of other operators. The separation property is "weaker" then the subformula property. As we will see, there are (classical) separable systems that don't have the subformula property. The converse is not true[45] : the separation property is a collorary of the subformula property.

Intuitionistic logic seems to fit all requirements. It has desirable properties: both the subformula and the separation property holds in LJ and in NJ. Classical logic appears to be more troublesome. Here one could point out the fact that LK, Gentzen's sequent calculus system for classical logic, has both properties. But, from a philosophical point of view, one could argue that it has a drawback: it is a multiple conclusion system[46].[47]

On the other hand, Natural deduction systems seems better suited for intuitionistic logic: some desirable properties are lost with standard formalizations of classical logic [48] and normalization procedures for intuitionistic logic are simpler.

This section contains: (I) a history of the separability theorem for intuitionistic logic, (II) a discussion about separability in natural deduction systems for classical logic, and (III) a discussion about different grades of analyticity.

2.1 **Separability in intuitionistic logic**

The first results regarding the separation property can be traced back to Gentzen (1934), Wajsberg, and Curry. Gentzen didn't mention the separation theorem explicitly, but, as Prawitz points out [49], "a similar theorem for the calculus of sequents is an immediate corollary of Gentzen's Hauptsatz". [50] The first person to state the theorem was Wasjberg in the 1938 article "Untersuchungen über den Aussagenkalkül" (WAJSBERG, 1938).

> "This work is mainly dedicated to proving that every consequence of M is derivable from those of its axiom groups which, besides implication, contain only those connectives that appear in the consequence in question." (WAJS-

[45] A counter example will be presented in section 5.

[46] It is not the objective of this dissertation to argue in favor or against multiple conclusion systems. But, the author would like to humbly point out his preference for single conclusion systems - they seem more "natural". As humans, we can only draw one conclusion at once. Then we draw another, and then another. But not at the same time! The inferential practice seems to be sequential.

[47] Gabbay and Gabbay's natural deduction system for classical logic is a natural deduction that has the subformula property. This is obtained by the addition of a structural rule that "mimics" LK's ability to make multiple conclusions at once. See (GABBAY; GABBAY, 2005)

[48] By "standard" one should understand formalizations using some form of *Double negation elimination, Reductio ad absurdum* or *Tertium non Datur*

[49] (PRAWITZ, 1965, pg.54)

[50] As it was mentioned, the subformula property entails the separation property

BERG, 1938) [51]

But, as indicated by Church and Horn [52], Wajsberg's "proof" [53] contains an error:

> "On page 142 it should have been pointed out that Wajsberg's paper, cited in footnote 211, contains an error that is not easily set right. However, the metatheorem that is stated in the next-to-last paragraph of the text on page 142, and a similar metatheorem for the formulation F" of intuitionistic functional calculus of first order, were proved by Curry in the Bulletin of the American Mathematical Society, vol. 45 (1939), pp. 288-293, and the proof is reproduced by Kleene in Introduction to Metamathematics.— Since Curry's proof depends on Gentzen's Hauptsatz for LJ, the remark should be made that it is not the use of Gentzen's Sequenzen but the Hauptsatz itself that is essential, as the Sequenzen can of course be eliminated by the definitions on page 165 (with m= 1 for the intuitionistic case), and the Hauptsatz therefore proved in a form that is directly applicable to formulations of the ordinary kind without Sequenzen (compare Curry, loc. cit., and Kurt Schutte in the Mathematische Annalen, vol. 122 (1950), pp. 47-65)." (CHURCH, 1957) pg.377

Curry was probably the first person to prove the theorem directly. In his own words:

> "This result appears to be generally known; but I am acquainted with no published proof of it. For partial results of the same nature see D. Hilbert and P. Bernays, Grundlagen der Mathematik, vol 1, 1934, p. 71; also I. Johansson, Compositio Mathematica, vol.4 (1937), p. 131. Added in proof: See also M. Wajsberg, Untersuchungen Uber den Aussagenkalkul von A . Heyting, WiadomoSci Matematyczne, vol. 46 (1938), pp. 45-101." Curry 1939 pg.289 (CURRY, 1939)

His result was obtained for Heyting's LHJ, an axiomatic intuitionistic calculus:

> "if we take the axioms of LHJ as a basis, that the schemes for implication follow from the axioms for implication only and that those for conjunction, negation, and the quantifiers, respectively, involve only the axioms for implication and those for the operation concerned." Curry 1939 (CURRY, 1939) pg.2

Prawitz formulated the theorem for intuitionistic and minimal logic in terms of natural deduction:

---

[51] Free translation from the author
[52] see Church 1957 pg. 377(CHURCH, 1957) and Horn 1962 pg.1 (HORN, 1962)
[53] "proof" is under quote here because, since it contains an error, it cannot be called a proof.

> "**Corollary IV.** (separation theorem) If Π is a normal deduction in I or in M of A from Γ, then the only inference rules that are applied in Π are the inference rules in I or in M for the logical constants that occur in A or in some formula of Γ" (PRAWITZ, 1965, pg.54)

The reader should notice a difference in Prawitz's formulation: instead of axioms, rules are mentioned. Also, there is no direct reference to implication.

2.2 Separability in classical logic

At first glance, one could think that there can be no separable natural deduction systems for classical logic. In fact, Julian Murzi [54] discusses an interesting theorem proved by Hughes Leblanc:

> "Theorem 2. (LEBLANC, 1966, pg.35) If either Double Negation Elimination or classical reductio (or some equivalent rule) are taken to partly determine the meaning of classical negation, then no complete natural deduction formalization of classical logic is separable." [55]

What Leblanc showed was that a whole class of formalizations of classical logic in natural deduction, namely, those involving negation, aren't separable [56]. But this does not mean that separable formalizations of classical logic aren't possible. Such an interpretation - that there is no separable formulation of classical logic in natural deduction - was given by Bendall:

> "certain facts pointed out by Leblanc (1966) as "shortcomings of natural deduction" cause trouble (and otherwise it is not clear why they should be called "shortcomings"). Namely, Leblanc shows, in effect, that no classically complete [natural deduction formalization of classical logic] is separable. So [...] the separation problem for such languages appears to be unsolvable." [57]

Leblanc's theorem, contrary to Bendall's interpretation, doesn't rule out the possibility of a separable classical system of natural deduction. The restriction is upon formalizations where the classicizing rule involves negation. Here are three different proofs of Peirce's axiom, in three different systems:

1. NJ + Double negation elimination

[54] see Murzi 2005 (MURZI, 2010) pg.199
[55] (LEBLANC, 1966, pg.35) ibid. Murzi 2005 (MURZI, 2010) pg.199
[56] Professor Jean-Baptiste Joinet pointed out that LeBlanc's theorem only holds for single conclusion Natural Deduction Systems
[57] Bendall ibid. Murzi. see (MURZI, 2010) pg.199

$$\dfrac{Dne\,\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^2 \quad [\neg p]^1}{\bot}}{q}}{p \to q}\,2 \quad [(p \to q) \to p]^3}{p} \quad [\neg p]^1}{\bot}}{\neg\neg p}\,1}{p}}{((p \to q) \to p) \to p}\,3$$

2. NJ + Reductio ad absurdum

$$\dfrac{Raa\,\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]^2 \quad [\neg p]^1}{\bot}}{q}}{p \to q}\,2 \quad [(p \to q) \to p]^3}{p} \quad [\neg p]^1}{\bot}}{p}\,1}{((p \to q) \to p) \to p}\,3$$

3. NJ + Tertium non datur [58]

$$\dfrac{Tnd\,\dfrac{p \lor \neg p \quad [p]^0 \quad \dfrac{\dfrac{\dfrac{\dfrac{[p]^2 \quad [\neg p]^1}{\bot}}{q}}{p \to q}\,2 \quad [(p \to q) \to p]^3}{p}}{p}\,0,1}{((p \to q) \to p) \to p}\,3$$

In order to prove Peirce's axiom, $(((p \to q) \to p) \to p)$ the negation of the proposition p must be assumed. Leblanc's theorem could be interpreted in the following way: Separable classical systems cannot be constructed through the addition of a new rule for negation to an intuitionistic system. Leblanc's theorem doesn't rule out the possibility of constructing a separable classical system through the addition of a rule that doesn't involve negation. In the following sections, we will discuss two ways out of this dilemma [59]: (1) one could enhance an intuitionistic natural deduction calculus through the addition of an implicational rule, or (2) one could enhance an intuitionistic natural deduction calculus through the addition of a structural rule.

[58] proof present in (PEREIRA et al., 2023) pg.12

[59] One could, having in mind Bendall's interpretation of Leblanc's theorem, formulate the following dilemma: A natural deduction system can be (i) separable or (ii) classical, but not both. As we will see, that's not the case. There are classical systems of natural deduction that are separable

2.3 Types of analyticity

Separability and subformula are desirable properties. But, it is not always the case that classical systems in natural deduction have them. In fact, there are systems that have none (in special, systems where the ”classicizing” rule is a negational rule), systems that are only separable (two examples involving implication will be discussed in this dissertation) and systems that have both properties (we will discuss two examples, both are obtained by the addition of structural rules).

Both the separation and the subformula property says something about the *analyticity* of proofs: the subformula property tells us that every information used in the proof is ”contained” in the conclusion, i.e, given a theorem θ, there is a proof of θ that only contains subformulas of θ. But, the separation property tells us that, given theorem θ, there is a proof of θ where only the rules involving the logical connectives present in θ will be used in the proof of θ. As it was mentioned, the separation theorem is weaker than the subformula principle. It only concerns logical connectives[60]:

Definition 2.3.1 *Separation property: A system S has the separation if, whenever Π is a Normal Deduction of A from Γ in S, then the only inference rules that are applied in Π are the inference rules for the logical constants that occur in A or in some formula of* Γ.

Definition 2.3.2 *Subformula property: A system S has the subformula property if, whenever $\Gamma \vdash_s A$, then there is a proof of A from Γ where only subformulas of A or a of some formula in Γ are used.*

The separation property and the subformula property allow us to talk about two senses of analyticity. The separation property speaks about containment in the following sense: if the proof of a formula A from Γ is separable, this means that every rule used in order to prove A from Γ is a rule of a logical operator that is present/contained in A or in some formula of Γ. The subformula property speaks of a stronger sense of containment, and thus, of a stronger sense of analyticity: if a proof of A from Γ respects the subformula property, this means that in every step of the proof, only subformulas of A or of Γ will be used. The subformula property tells us that every information needed in order to prove A from Γ is already contained in A or in Γ, i.e, only subformulas of A of of some formula of Γ will be used in the derivation of A from Γ. It is a stronger sense of containment than that of the separation property.

The idea, developed by Murzi, following the works of Tennant (1997) (TENNANT, 1997) and Shapiro (1998) (SHAPIRO, 1998) is that there are *grades of analyticity* [61]. Murzi divides inferences into:

---

[60] definitions below taken from (MURZI, 2010), pg.188
[61] expression taken from Shapiro (1998, pg.611) (SHAPIRO, 1998)

Definition 2.3.3 *"Strictly Analytic: an inference* $\Gamma \vdash A$ *is strictly analytic if A can be derived from* Γ *by means of a proof in which only operational rules* [62] *for the logical operators occurring in A or* Γ *are used."*

Definition 2.3.4 *"Ultra Strictly Analytic: an inference* $\Gamma \vdash A$ *is ultra strictly analytic if A can be derived from* Γ *by means of a derivation satisfying the subformula property."* [63]

This definition concerns inferences, but one could extend it to systems:

Definition 2.3.5 *Strictly Analytic System: a system* S *is Strictly Analytic if for every* $\Gamma \vdash_S A$ *there is a derivation* Π *of A from* Γ *in* S *where only operational rules for the logical operators occurring in A are used.*

Definition 2.3.6 *Ultra Strictly Analytic System: a system* S *is Ultra Strictly Analytic if for every* $\Gamma \vdash_S A$ *there is a derivation* Π *of A from* Γ *in* S *that satisfies the subformula property.*

The question remains: is it possible to formulate a separable formalization of classical logic in natural deduction? The short answer is yes. In the following chapters we will see two types of formalization of classical logic in natural deduction: (1) through implicational rules and (2) through structural rules. The first road results in Strictly analytic systems, the second in Ultra Strictly Analytic systems.

---

[62] An operational rule is a rule that contains a logical operator (such as $\wedge$, $\rightarrow$, $\vee$ etc...). Operational rules should be contrasted with structural rules, i.e, rules that contain no logical operators.

[63] see (Murzi pg.188 (MURZI, 2010)) .

3 THE STRUCTURAL ROAD

The aim of this chapter is to present ways of obtaining a separable classical propositional calculus from NJ through structural changes. Two systems will be explored: the first one (i) was developed by Michael and Murdoch Gabbay (GABBAY; GABBAY, 2005) and the second one (ii) was developed by Julian Murzi (MURZI, 2010). Both of them are $Ultra\ Strictly\ Analytic$ systems.

Gabbay and Gabbay's system is in a lot of ways similar to Gentzen's sequent calculus LK: "it maintains the subformula property" and it "preserves the formal structure of deductions", meaning that, deductions in NJ are still deductions in the system. Their strategy is to enrich NJ with a structural rule that corresponds, in a lot of ways, to the LK property of allowing multiple conclusions.

Murzi's system follows a different strategy. His ideia is to (i) use Schroeder-Heister's [64] notion of higher order rules. This way, assumptions of the form $A \rightarrow B$ can be formulated as higher order assumptions of the form A/B. The second step is to treat $\perp$ not as an atom of the language, but as a punctuation mark. The allowance of higher order rules toghether with this 'shift of perspective' concerning $\perp$ is what makes Murzi's system an $Ultra\ Strictly\ Analytic$ system.

3.1 NJ + $Restart$ rule

3.1.1 Structural rules in Natural Deduction

One way of obtaining a classical separable system from NJ is by the addition of a structural rule. A rule is said to be structural when it doesn't deal with any logical operators. In sequent calculus, structural rules such as:

- Left Contraction

$$\frac{A, A, \Gamma \vdash \Theta}{A, \Gamma \vdash \Theta}$$

Are written out just like the other rules. In Natural Deduction systems structural properties are a bit more subtle. They aren't explicitated as rules. The following derivation for example,

[64] (SCHROEDER-HEISTER, 1984)

$$\cfrac{\cfrac{\cfrac{\cfrac{[A]^1 \qquad [A \to (A \to B)]^2}{A \to B} \qquad [A]^1}{B}}{A \to B}\,1}{(A \to (A \to B)) \to (A \to B)}\,2$$

Is allowed. But, the derivation above only works if more than one hypothesis can be discharged at once. This is a structural property and it corresponds to the Left Contraction structural rule. Natural Deduction systems have structural properties similar to sequent calculus.

3.1.2 Gabbay and Gabbay's Rule

In 2005, Michael and Murdoch Gabbay, proposed a structural rule that "can neatly [65] and cheaply [66] capture classical and intermediate logics." (GABBAY; GABBAY, 2005).

Their proposal is to extend intuitionistic first order logic with **classical restart**, a Natural Deduction structural rule:

$$\frac{A}{B}\,Restart^*$$

But, there is a side condition:

- *Below every occurrence of restart from A to B, there is (at least) one occurrence of A.* (GABBAY; GABBAY, 2005) pg.2

The realization of the side condition above is marked by the $*$ sign. The $*$ sign justifies the application of restart:

$$\begin{array}{c}\dfrac{A}{B}\,Restart^* \\ \vdots \\ A^* \end{array}$$

Surprisingly enough, the addition of classical restart to NJ yields a classical natural deduction system that has the following properties (i) separability, (ii) subformula:

- *"Methods of obtaining a calculus for classical logic that satisfies a normal form theorem are known. For example in [Stalmarck, 1991, Prawitz, 1965] the rule*

[65] By "neatly" they mean (i) that it preserves proof normalization i.e the property of reducing deductions to a normal form and (ii) that it preserves the subformula property (which entails the separation property)
[66] By "cheaply" they mean the property of preserving the formal structure of derivations.

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots \\ \bot\end{array}}{A}\ (\text{PIP})$$

is used. This rule may be considered as an elimination rule for $\bot$, it is also known as the principle of indirect proof (PIP). We confine discussion of this rule to a footnote because it is not purely structural, it cannot be formulated unless at least one of $\bot$ or $\neg$ is in the language. So for example we cannot use it to formulate the implication-only fragment of classical propositional logic which should have $((A \to B) \to A) \to A$ as a theorem independently of whether we have negation or $\bot$ in the language as well. We are interested here in the properties of Restart as a purely structural rule that makes the difference between intuitionistic, classical and intermediate logics." (GABBAY; GABBAY, 2005) pg.3

Here are two proofs of Peirce's formula, one in Sequent Calculus, another in Natural Deduction:

- Peirce's formula proof in LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A \vdash A, B}}{\vdash A, (A \to B)} \qquad A \vdash A}{(A \to B) \to A \vdash A, A}}{(A \to B) \to A \vdash A}}{((A \to B) \to A) \to A}$$

- Peirce's formula proof in NJ + Restart [67]

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^1}{B}\ (\text{Restart})^*}{A \to B}\ 1 \qquad [(A \to B) \to A)]^2}{A^*}}{((A \to B) \to A) \to A}\ 2$$

Allowing restart in a Natural deduction system yields the same effect as allowing multiple conclusions in a sequent calculus system. In the LK proof, B is obtained using weakning right, while in the NJ + Restart proof, B is obtained through restart.

[67] Proof taken from (GABBAY; GABBAY, 2005)

3.2 **Murzi's** NCP^+

Another way of obtaining a separable classical system from NJ is by the addition of a higher-order rule. This system was developed by Murzi in (MURZI, 2010). It is an assertion based, single-conclusion, separable system for classical logic called NCP^+.

Similar to the restart rule, this system also entails the subformula property.

The NCP^+ system consists of

(i) the standard intuitionistic rules for $\wedge - I, \wedge - E, \rightarrow -I, \rightarrow -E, \neg - I, \neg - E$

(ii) two non-standard rules for disjunction:

$$\dfrac{\begin{array}{c}[A/\bot] \qquad [B/\bot] \\ \vdots \\ \bot\end{array}}{A \vee B}$$

and:

$$\dfrac{A \vee B \qquad [A/\bot] \qquad [B/\bot]}{\bot}$$

(iii) Two structural rules:

$$\dfrac{\begin{array}{c}[A]^a \\ \vdots \\ B\end{array}}{(A/B)}\, a$$

$$\dfrac{(A/B) \qquad A}{B}$$

(iv) A higher order version of the *Reductio ad Absurdum* rule [68]:

$$\dfrac{\begin{array}{c}[A/\bot]^i \\ \vdots \\ \bot\end{array}}{A}\, i$$

[68] Murzi points out that if $\bot$ is treated as a punctuation mark, and not as an atom of the language, then, the higher order version of Raa turns out to be a structural rule. The ideia that $\bot$ can be treated as a punctuation mark comes from Tennant 1999 and Rumfitt 2000

NCP+ is a separable formalization of classical logic only if [69]:

(a) One accepts Tennant's and Rumfitt's suggestion that $\perp$ is best treated as a logical punctuation sign.

(b) One accepts Schroeder-Heister's invitation to regard higher-order rules as legitimate.

3.2.1 $\perp$ as a punctuation mark

Murzi enumerates three different views about absurdity: Prawitz's, Dummet's and Tennant/Rumfitt's [70]. This discussion is beyond the scope of this work. We will restrict the discussion to a mere description of each position.

Prawitz's suggestion is that there is no introduction rule for$\perp$:

> *the introduction rule for $\perp$ is empty, i.e. it is the rule that says that there is no introduction whose conclusion is $\perp$.*(MURZI, 2020, pg.11)

Dummett suggests that the following infinitary rule defines the meaning of $\perp$:

$$\frac{p_1 \qquad p_2 \qquad p_3 \qquad \ldots}{\perp}$$

Where p_n is an atom of the language. In Murzi's view, "Both Prawitz's and Dummett's accounts are problematic":

> *"Dummett's rule is non recursive and makes the meaning of $\perp$ dependent on the expressiveness of one's language. After all, it may be argued that atoms need not be in general incompatible. As for Prawitz's account of $\perp$, the very thought that $\perp$ has content makes the meaning of negation dependent on the meaning of absurdity, and hence violates the orthodox inferentialist's demand for purity."*

He presents an alternative view, following Tennant and Rumfitt. The idea is to treat $\perp$ as a ponctuation sign:

> *an occurrence of '$\perp$' is appropriate only within a proof [...] as a kind of structural punctuation mark. It tells us where a story being spun out gets tied up in a particular kind of knot—the knot of a patent absurdity, or self contradiction.* (TENNANT, 1999, pg.204) ibid. (MURZI, 2020, pg.12)

[69] see (MURZI, 2010)pg.264
[70] see Tennant 1999 (TENNANT, 1999) and Rumfitt 2000 (RUMFITT, 2000)

3.2.2 Higher order rules

Schroeder-Heister's idea of higher order rules was presented in the 1984 paper (SCHROEDER-HEISTER, 1984) called "A natural extension of natural deduction". The opening paragraph is as follows:

- *"One of the main ideas of calculi of natural deduction, as introduced by Jaśkowski and Gentzen, is that assumptions may be discharged in the course of a derivation. As regards sentential logic, this conception will be extended in so far as not only formulas but also rules may serve as assumptions which can be discharged."* (SCHROEDER-HEISTER, 1984) pg.1

  As Murzi puts it,

- *"Natural deduction systems involve rules, such as arrow introduction, which allow one to discharge assumptions. [...] But if assumptions just are ad hoc axioms, one should also be free to use ad hoc rules in the context of a derivation."*(MURZI, 2020) pg.12

  The idea is to allow one to make assumptions of rules that may be discharged, not just of formulas.

3.2.3 NCP^+ Normalization

Murzi (MURZI, 2020) proves the subformula property for NCP+. As we've seen, the subformula property entails the separability property. Murzi's proof consists in a generalization of Prawitz's original proof for C'. The reductio rule is problematic. It breaks the subformula property. Murzi's trick is to "externalize" negation.

The classical reductio rule:

$$\dfrac{\begin{array}{c}[\neg A]\\ \vdots \\ \bot\end{array}}{A}$$

becomes a higher order structural rule:

$$\dfrac{\begin{array}{c}[A/\bot]\\ \vdots \\ \bot\end{array}}{A}$$

In fact, they are interderivable. [71]:

[71] see (MURZI, 2020) Murzi, 2020, pg.17, Lemma 13

- First, prove that $A/\bot$ follows from $\neg A$

$$\dfrac{\dfrac{A \qquad [\neg A]}{\bot}}{A/\bot}$$

- Then, it is enough to show that $\neg A$ follows from $A/\bot$:

$$\dfrac{\dfrac{A \qquad [A/\bot]}{\bot}}{\neg A}$$

Prawitz's procedure ensures that every application of the *Reductio ad Absurdum* rule is atomic. In a similar fashion, Murzi's proof reduces every application of *Reductio ad Absurdum* to atomic formulas. Since *Reductio ad Absurdum* is treated as a structural rule, assuming that $\bot$ is a punctuation mark, this version of the Reductio respects the subformula property. Murzi also finds a way to deal with $\vee$. Here are his transformations:

1. $\neg$

$$\begin{array}{c} [\neg A/\bot]^1 \\ \Sigma \\ \dfrac{\bot}{\neg A}\,1 \end{array}$$

Tranforms into

$$\begin{array}{c} \dfrac{\dfrac{[A]^1 \qquad [\neg A]^2}{\bot}}{(\neg A/\bot)}\,2 \\ \Sigma \\ \dfrac{\bot}{\neg A}\,1 \end{array}$$

2. $\wedge$

$$\begin{array}{c} [A \wedge B/\bot]^1 \\ \Sigma \\ \dfrac{\bot}{A \wedge B}\,1 \end{array}$$

Transforms into

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[A\wedge B]^1}{A}\qquad [A/\bot]^2}{\bot}\,1}{(A\wedge B/\bot)}\\ \Sigma\\ \bot\end{array}}{A}\,2 \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[A\wedge B]^3}{B}\qquad [B/\bot]^4}{\bot}\,3}{(A\wedge B/\bot)}\\ \Sigma\\ \bot\end{array}}{B}\,4}{A\wedge B}$$

3. $\to$

$$\dfrac{\begin{array}{c}[A\to B/\bot]^1\\ \Sigma\\ \bot\end{array}}{A\to B}\,1$$

transforms into:

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[A]^1\qquad [A\to B]^2}{B}\qquad [B/\bot]^3}{\bot}}{(A\to B/\bot)}\,2\\ \Sigma\\ \bot\end{array}}{B}\,3}{A\to B}\,1$$

4. $\vee$

$$\dfrac{\begin{array}{c}[A\vee B/\bot]^1\\ \Sigma\\ \bot\end{array}}{A\vee B}\,1$$

transforms into:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{[A\vee B]^1\qquad \dfrac{[A]^2\qquad [A/\bot]^3}{\bot}\qquad \dfrac{[B]^4\qquad [B/\bot]^5}{\bot}}{\bot}\,2,4}{[A\vee B/\bot]}\,1\\ \Sigma\\ \bot\end{array}}{A\vee B}\,3,5$$

Having defined these procedures, Murzi is able to prove a normalization procedure for $NCP+$ that entails the following corollary;

> **Corollary 30 (Subformula property)** *Each formula occurring in a normal deduction* Π *of* γ *from* Γ *is a subformula of* γ *or of one of the formulae in* Γ.
> Prawitz (1965, pp. 42–43) proves this result for his own formalization of CPL, which includes the rules for $\wedge$, $\rightarrow$, and CR [Reductio ad Absurdum], and where ¬ A is defined as A $\rightarrow \perp$. In Prawitz's system, the theorem holds for every formula in Π, 'except for assumptions discharged by applications of CR and for occurrences of $\perp$ that stand immediately below such assumptions'. Prawitz's proof carries over to Ncp+, this time without exceptions. Informally, this can be shown by considering, in the new Ncp+ setting, the exceptions to Prawitz's original theorem, viz. that (i) assumptions discharged by applications of CR and (ii) occurrences of $\perp$ that stand immediately below such assumptions may not be subformulae of either γ or some of the formulae in Γ. Concerning (i), we then notice that it is a consequence of Prawitz's theorem that, if $B/\perp$ is an assumption discharged by CRhl in a normal deduction of A from Γ, then B is a subformula of A or of some subformula of Γ. As for (ii), the problem disappears as soon as we treat $\perp$ as a logical punctuation sign. For a fuller proof, we first order branches according to the following definition, still following and generalising Prawitz's original proof. (MURZI, 2020, pg.21)

4 THE IMPLICATIONAL ROAD

The aim of this chapter is to explore ways of obtaining a separable classical system through the addition of an implicational rule to NJ.

4.1 Peirce's rule

Peirce's axiom is at the heart of our discussion. It is an implicational formula that's only provable in classical logic:

- $((p \to q) \to p) \to p$

Curry was the first to propose a "purely implicative elimination rule emerging from the famous Law of Peirce, a purely implicative theorem, being provable only within classical logic." (ZIMMERMANN, 2002, pg.1):

$$\begin{array}{c} [A \to B]^1 \\ \Pi \\ \dfrac{A}{A}\,1 \end{array}$$

In this section, a normalization procedures for Peirce's rule will be explored.

4.1.1 Pereira, Haeusler, Costa and Sanz

In 2010 Pereira, Haeusler, Costa and Sanz published a paper under the name "A New Normalization Strategy for the Implicational Fragment of Classical Propositional Logic" (PEREIRA et al., 2010). The motivation of the paper is the following: present a new normalization strategy for the system composed of the usual intuitionistic I and E rules for $\to$ and Peirce's rule:

- $NPimp$

$$\begin{array}{c} [A]^1 \\ \vdots \\ \dfrac{B}{A \to B}\,1 \end{array} \qquad \dfrac{A \quad A \to B}{B} \qquad \begin{array}{c} [A \to B]^1 \\ \vdots \\ \dfrac{A}{A}\,1 \end{array}$$

Their normalization strategy is similar to Seldin's [72]. The idea is to postpone the applications of the classicizing rule in question. In Seldin's case, $\perp_c$, and in [Pereira, Haeusler,

[72] see (SELDIN, 1989)

Costa and Sanz], Peirce's rule. As a result, every canonical proof will be separated into an intuitionistic part and a classical part, in such a way that there is no applications of intuitionistic reasoning bellow the first application of classical reasoning [73]. In Seldin's case, the applications of $\bot_c$ can be reduced to, at most, one application in such a way that it is the last rule applied in the proof: every classical proof Π_c of A [74]

$$\begin{array}{c} \Pi_c \\ A \end{array}$$

Can be transformed into a proof Π^* of A where: (i) Π^* is divided into an intuitionistic and a classical part where (ii) the classical part contains at most one application of the *Reductio ad Absurdum* rule. Π^* has the following form:

$$\begin{array}{c} [\neg A]^1 \\ \Pi_i \\ \dfrac{\bot}{A}\,1 \end{array}$$

Where Π_i is intuitionistic. See (GUERRIERI; NAIBO, 2019).

Pereira, Haeusler, Costa and Sanz's strategy is similar. The objective is to postpone the applications of Peirce's rule in such a way that every classical proof Π_c of A:

$$\begin{array}{c} \Pi_c \\ A \end{array}$$

Can be transformed into a proof Π^* of A where (i) Π^* is divided into an intuitionistic and a classical part and (ii) the classical part consists in a series of applications of Peirce's rule:

$$\begin{array}{c} [A \to b_1]^1, ..., [A \to b_n]^n \\ \Pi_i \\ \dfrac{A}{A}\,1 \\ \vdots \\ \dfrac{A}{A}\,n \end{array}$$

In the fragment $[\to]$, it is not possible to reduce the applications of Peirce to at most one. To do this one must enrich the system with conjunction. Then, instead of assuming $A \to b_1, ..., A \to b_n$, we simply assume $A \to (b1 \wedge ... \wedge bn)$ n times. See (PEREIRA et al., 2010):

[73] The intuitionistic region is the 'top' region of the proof and the classical part is the 'bottom' region
[74] for more information on Seldin's normalization strategy see: (GUERRIERI; NAIBO, 2019) and (SELDIN, 1989)

$$\dfrac{\dfrac{\dfrac{[A \to (b_1 \wedge ... \wedge b_n]^1 \qquad [A]}{b_1 \wedge ... \wedge b_n}}{b_1}}{A \to b_1} \qquad ... \qquad \dfrac{\dfrac{\dfrac{[A \to (b_1 \wedge ... \wedge b_n]^1 \qquad [A]}{b_1 \wedge ... \wedge b_n}}{b_n}}{A \to b_n}$$

$$\Pi_i$$

$$\dfrac{A}{A}\,1$$

With conjunction added to the system, the classical part will have at most one applications of the P-rule.

4.1.1.1 Another version of Glivenko's Theorem

Pereira, Haeusler, Costa and Sanz's normalization strategy entails an interesting collorary. It is a new version of Glivenko's theorem[75]:

Theorem 4.1.1 ($Glivenko^*$) *Let $p_1, ..., p_n$ be the set of atomic formulas in A. Then, $\vdash_{NPimp} A$ if and only if $\vdash_{NIimp} (A \to p_1) \to ((A \to p_2)...((A \to p_n) \to A)...)$.*

The theorem above guarantees that if there is a classical proof of A, then there is an intuitionistic derivation of A, from the open premises $A \to p_1, A \to p_2 ... A \to p_n$ [76]. This theorem can thus be stated as:

If $\vdash_{NPimp} A$, then $A \to p_1, A \to p_2, ..., A \to p_n \vdash_{NIimp} A$

4.1.1.2 $NP_{imp\bot}$

Now, let's add $\bot$ to NP_{imp}. The resulting system is complete in relation to classical logic. All logical constants can be defined:

- negation $\neg A := A \to \bot$
- disjunction $A \vee B := (A \to B) \to B$
- conjunction $A \wedge B := (A \to (B \to \bot)) \to \bot$

[75] The $NIimp$ system referenced in the theorem below is a system composed of the $I \to$ rule and the $E \to$ rule. $NIimp$ stands for Natural Intuitionistic Implicational system

[76] This Glivenko variant was initially thought for a classical system composed of NJ + Peirce, by (PEREIRA et al., 2010)

4.1.1.3 $\perp$ reduction

Consider the following derivation Π:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[\perp \to P_1]^1, ..., [\perp \to P_k]^k \\ \Pi_i \\ \perp\end{matrix}}{\perp}\,1}{\vdots}}{\dfrac{\perp}{\perp}\,k}}{B}$$

Then, apply the following reduction procedure:

- **$[P_\perp]$ reduction**

$$\begin{matrix} \dfrac{\dfrac{[\perp]}{P_k}}{\perp \to P_k} & [\perp \to P_1]^1, ..., [\perp \to P_{k-1}]^{k-1} \\ & \Pi_i \\ & \dfrac{\perp}{\perp}\,p_1 \\ & \vdots \\ & \dfrac{\perp}{\perp}\,p_{k-1} \\ & \overline{B} \end{matrix}$$

Each application of $[P_\perp]$ reduction reduces by 1 the number of applications of Peirce's rule. The derivation above can be reduced to:

$$\begin{matrix} \dfrac{\dfrac{[\perp]}{P_1}}{[\perp \to P_1]} & .... & \dfrac{\dfrac{[\perp]}{P_k}}{[\perp \to P_k]} \\ & \Pi_i & \\ & \dfrac{\perp}{B} & \end{matrix}$$

4.2 Hosoi's sequent calculus system

In 1966 Tsutomu Hosoi, from the University of Tokyo, published a paper under the name "The Separation Theorem on the classical system" (HOSOI, 1966), in which he presents a system for first order logic consisting of LJ + $(A \cup B), (A \to C), (B \to C) \vdash C$. [77]

The addition of this derivability relation[78] to LJ results in a system with surprising properties: (I) It allows one to define the right side of sequents in LK as a single formula using only implication, (II) it is a classical single conclusion sequent calculus system and (III) it has the separation property, which states that: *For a provable formula in the system, there is a proof in which only the axioms for implication and the axioms for the other logical symbols actually appearing in the formula are used.*

His objective is to investigate what he calls intuitionistic *foundational* systems. An intuitionistic system can be called *foundational* if one can construct a **separable classical system** from it by adding one new axiom. Hosoi proves that Genzen's LJ system is a foundational system.

The construction of LK from LJ, for example, does not guarantee for LJ the *foundational* status since LK is constructed from LJ by means of a structural change, viz, the allowance of multiple conclusions.

4.2.1 Hosoi's separability proof

Hosoi's aim is to prove that his system is a classical separable system, showing that LJ is *foundational*. Here is a brief outline of his proof:

(1). First, he demonstrates that the new axiom is provable in the classical system LK, which implies that Hosoi's system is a subsystem of LK:

[77] Where $\cup$ is not a primitive sign. $A \cup B$ stands for $((A \to B) \to B)$.
[78] From now on the derivability relation $(A \cup B), (A \to C), (B \to C) \vdash C$ will be referred to as Hosoi's rule.

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\LeftLabel{wr}
\UnaryInfC{$A \vdash A, C$}
\LeftLabel{wr}
\UnaryInfC{$A \vdash A, B, C$}
\LeftLabel{$\rightarrow r$}
\UnaryInfC{$\vdash A, C, A \rightarrow B$}
\AxiomC{$B \vdash B$}
\LeftLabel{$\rightarrow l$}
\BinaryInfC{$(A \rightarrow B) \rightarrow B \vdash A, B, C$}
\AxiomC{$C \vdash C$}
\LeftLabel{$\rightarrow l$}
\BinaryInfC{$(A \rightarrow B) \rightarrow B, A \rightarrow C \vdash C, C, B$}
\AxiomC{$C \vdash C$}
\LeftLabel{$\rightarrow l$}
\BinaryInfC{$(A \rightarrow B) \rightarrow B, A \rightarrow C, B \rightarrow C \vdash C, C, C$}
\LeftLabel{cr}
\UnaryInfC{$(A \rightarrow B) \rightarrow B, A \rightarrow C, B \rightarrow C \vdash C, C$}
\LeftLabel{cr}
\UnaryInfC{$(A \rightarrow B) \rightarrow B, A \rightarrow C, B \rightarrow C \vdash C$}
\end{prooftree}
$$

Then, he demonstrates that LK is a subsystem of Hosoi's. To do this, he must show that $\neg\neg A \rightarrow A$ is derivable in his system. This can be done by substituting A,B, and C for A, a contradiction and A respectively in the axiom:

$$(A \rightarrow B) \rightarrow B, A \rightarrow C, B \rightarrow C \vdash C$$

$$(A \rightarrow \bot) \rightarrow \bot, A \rightarrow A, \bot \rightarrow A \vdash A$$

The proof consists in a series of cuts:

$$
\begin{prooftree}
\AxiomC{$\bot \vdash$}
\UnaryInfC{$\bot \vdash A$}
\UnaryInfC{$\vdash \bot \rightarrow A$}
\AxiomC{$A \vdash A$}
\UnaryInfC{$\vdash A \rightarrow A$}
\AxiomC{$((A \rightarrow \bot)) \rightarrow \bot), (A \rightarrow A), (\bot \rightarrow A) \vdash A$}
\LeftLabel{cut}
\BinaryInfC{$((A \rightarrow \bot)) \rightarrow \bot), (\bot \rightarrow A) \vdash A$}
\LeftLabel{cut}
\BinaryInfC{$((A \rightarrow \bot)) \rightarrow \bot) \vdash A$}
\end{prooftree}
$$

So, Hosoi's system is equivalent to LK.

(2). The second step is to prove the following theorems using only implicational axioms:

1. $A \vdash (A \cup B)$
2. $B \vdash (A \cup B)$
3. $(A \cup (B \cup C)) \vdash ((A \cup B) \cup C)$
4. $(A \cup B) \vdash (B \cup A)$
5. $(A \cup A) \vdash A$

The theorems above guarantee that every sequent in LK with multiple conclusions $\Gamma \vdash \delta_1, \delta_2, ..., \delta_n$ can be translated into a single conclusion sequent of the following form $\Gamma \vdash \delta_1 \cup \delta_2 \cup ... \cup \delta_n$. The theorems above allow Hosoi to define $\Gamma \vdash \delta_1, \delta_2, ..., \delta_n$ as $\Gamma \vdash \delta_1 \cup \delta_2 \cup ... \cup \delta_n$ in his system. Hosoi's axiom corresponds to the definition of classical disjunction via implication.

This gives us an intuition of why property (II) holds. Since every multiple conclusion sequent in LK can be translated into a single conclusion sequent in Hosoi's system, the result is that Hosoi's system bypasses the single conclusion restriction. Even though only single conclusion sequents are allowed, it's single conclusion sequents behave just as if they were multiple conclusion sequents.

(3). The next step is to prove that the structural rules of LK are derivable in Hosoi's system. Right Weakning follows from axioms 1. $A \vdash (A \cup B)$ and 2. $B \vdash (A \cup B)$. Right Contraction follows from 5. $A \cup A \vdash A$. Permutation follows from 3. $(A \cup (B \cup C)) \vdash ((A \cup B) \cup C)$ and 4. $(A \cup B) \vdash (B \cup A)$:

$$\frac{\Gamma \vdash \Theta}{\Gamma \vdash \Theta, A}$$

$$\frac{\Gamma \vdash \Theta, A, A}{\Gamma \vdash \Theta, A}$$

$$\frac{\Gamma \vdash \Theta, A, B, \Lambda}{\Gamma \vdash \Theta, B, A, \Lambda}$$

Usually, the inferences above would be structural inferences (weakening, contraction, permutation, and cut, respectively). But, since $A, B, ..., C$ is defined as $A \cup B \cup ... \cup C$, they lose the structural status.

(4). Then Hosoi shows that every proof in LK can be transformed into a proof in his system.

(5). From the steps outlined above, we have the following proof of the separability theorem:

> *"Suppose a formula D is provable in our [Hosoi's] system. Then, there is an LK-proof for the sequent $\rightarrow D$ since our system is equivalent to LK. Moreover, we can think, owning to Gentzen's Hauptsatz on LK, that the LK-proof for the sequent contains no cut. Then, we transform our proof into that of our system as was defined in 6 [the transformation rules of Hosoi's system to LK]. If the end sequent has at most one formula in the succedent, the endsequent is not changed by the transformation. So, the endsequent $\rightarrow D$ remains the same. It is also easily seen from the definition of the transformation that in the transformation of each inference, only implicational axioms and axioms for the logical symbols that concern the inference in question are used. And since in the LK proof without cut only those axioms for the symbols that actually appear in the endsequent are used besides structural axioms, the proof of our system, which is obtained now uses only implicational axioms and those for the symbols that appear in D. The property concerning the positive and the negative appearances of logical symbols are easily seen from $(A \cup (B \cup C)) \rightarrow ((A \cup B) \cup C)$ and $(A \cup B) \rightarrow (B \cup A)$. So the theorem is proved in our system"* (HOSOI, 1966)

4.3 Hosoi's natural deduction correspondent

The aim of this section is to show, via Hosoi's axiom, that NJ and some of it's fragments are *foundational*. Because we are working with Natural Deduction systems, we will use the notion of **rule** instead of **axiom**:

Definition 4.3.1 *A natural deduction system is **foundational** iff one can construct a separable classical system from it by adding one new operational rule.*

Murzi's system and Gabbay and Gabbay's (GABBAY; GABBAY, 2005) system are both classical separable systems, but this is done through structural changes. They cannot give NJ the *foundational* status. In this chapter, we will explore the behavior of Hosoi's Axiom in a Natural Deduction setting. As we will see, the addition of Hosoi's rule to NJ results in a classical separable system. NJ is foundational.

Hosoi's rule [Hr] [79]

$$\dfrac{(A \to B) \to B \qquad \begin{matrix}[A]^n \\ \Pi_1 \\ C\end{matrix} \qquad \begin{matrix}[B]^m \\ \Pi_2 \\ C\end{matrix}}{C}\; n, m$$

The aim of the following sections is to build a separable system for classical propositional logic in natural deduction. First, we will work only with the $<\to>$ fragment. The addition of Hosoi's rule to the usual introduction and elimination rules for $\to$ results in a system that is complete in relation to the implicational fragment of classical propositional logic. Then, we will add $\perp$ to the language. This addition results in a complete system in relation to classical propositional logic. Finally, we will work with all the logical operators and show that the addition of Hosoi's rule to NJ results in a separable natural deduction system for propositional classical logic. The normalization strategy that is employed is heavily inspired in (PEREIRA et al., 2010). The idea is to permute the applications of Hosoi's rule with the usual intuitionistic rules of NJ, in such a way that every deduction Π in normal form is structured in the following way:

$$\begin{matrix}\Pi_i \\ \Pi_c\end{matrix}$$

Where Π_i is an intuitionistic derivation and Π_c consists in a series of applications of the Hosoi rule. Having done that, the intuitionistic part of the derivation can be normalized using the usual normalization techniches for NJ.

[79] In November of 2023 professor Luiz Carlos Pereira presented the natural deduction version of Hosoi's Axiom in his class.

4.3.1 [→] fragment

The usual introduction and elimination rules for $\to$ aren't strong enough to cover the complete behavior of classical implication. In Prawitz's C system and in Gentzen's NK, Peirce's formula $((A \to B) \to A) \to A$ can't be proven without the use of a $\neg$ rule:

- **NK**

$$
\begin{prooftree}
\AxiomC{$[A]^1$}
\AxiomC{$[\neg A]^2$}
\BinaryInfC{$\bot$}
\UnaryInfC{B}
\RightLabel{1}
\UnaryInfC{$A \to B$}
\AxiomC{$[(A \to B) \to A]^3$}
\BinaryInfC{A}
\AxiomC{$[\neg A]^2$}
\BinaryInfC{$\bot$}
\RightLabel{2}
\UnaryInfC{$\neg\neg A$}
\UnaryInfC{A}
\RightLabel{3}
\UnaryInfC{$((A \to B) \to A) \to A$}
\end{prooftree}
$$

- **C**

$$
\begin{prooftree}
\AxiomC{$[A]^1$}
\AxiomC{$[\neg A]^2$}
\BinaryInfC{$\bot$}
\UnaryInfC{B}
\RightLabel{1}
\UnaryInfC{$A \to B$}
\AxiomC{$[(A \to B) \to A]^3$}
\BinaryInfC{A}
\AxiomC{$[\neg A]^2$}
\BinaryInfC{$\bot$}
\RightLabel{2}
\UnaryInfC{A}
\RightLabel{3}
\UnaryInfC{$((A \to B) \to A) \to A$}
\end{prooftree}
$$

In NK, one needs to use the rule of *Double Negation Elimination*, and in C, one needs to use the rule of *Reductio ad Absurdum* is order to to prove Peirce's formula. C and NK aren't separable for this reason: The implicational rules alone aren't enough to prove the implicational theorems of these systems. In both of them, we must assume $\neg A$ and then discharge the negation.

But, adding Hosoi's rule to the usual $\to$ rules results in a complete system with respect to the classical fragment of [→]. As discussed in the last chapter, Peirce's rule, when added to the intuitionistic fragment of $\rightarrow$ also results in a complete system with respect to the classical fragment of $[\rightarrow]$[80]. This section focuses on Hosoi's rule.

4.3.1.1 $NHimp$ system

Consider a Natural Deduction system containing only $[\rightarrow]$ and the following rules:

- $I\rightarrow$

$$\dfrac{\begin{array}{c}[A]^n\\ \vdots\\ B\end{array}}{A \rightarrow B}\, n$$

- $E\rightarrow$

$$\dfrac{A \qquad A \rightarrow B}{B}$$

- $H - Rule$ (Hosoi's rule)

$$\dfrac{((A \rightarrow B) \rightarrow B) \qquad \begin{array}{c}[A]^n\\ \Pi_1\\ C\end{array} \qquad \begin{array}{c}[B]^m\\ \Pi_2\\ C\end{array}}{C}\, n, m$$

Let's call this system $NHimp$ (Natural Hosoi's implicational logic). $NHimp$ is strong enough to define disjunction:

The introduction rules $I\vee$:

$$\dfrac{A}{A \vee B}$$

[80] For more information on Peirce's rule, viz. (PEREIRA et al., 2010) and (ZIMMERMANN, 2002)

and

$$\frac{B}{A \vee B}$$

And the elimination rule $E\vee$:

$$\cfrac{A \vee B \qquad \begin{matrix}[A]^1 \\ \vdots \\ C\end{matrix} \qquad \begin{matrix}[B]^2 \\ \vdots \\ C\end{matrix}}{C}\ 1,2$$

Can be translated into $NHimp$ using only $\rightarrow$ rules [81].

- $I\vee$ left

$$\cfrac{\cfrac{A \qquad [A \rightarrow B]^1}{B}}{(A \rightarrow B) \rightarrow B}\ 1$$

- $I\vee$ right

$$\frac{B}{(A \rightarrow B) \rightarrow B}$$

- $E\vee$

  In the case of $E\vee$, the translation is pretty straight foward:

$$\cfrac{(A \rightarrow B) \rightarrow B \qquad \begin{matrix}[A]^1 \\ \Pi_1 \\ C\end{matrix} \qquad \begin{matrix}[B]^2 \\ \Pi_2 \\ C\end{matrix}}{C}\ 1,2$$

[81] The procedures for $I\vee$ are identical to Zimmerman's procedure in (ZIMMERMANN, 2002)

4.3.1.2 Derivability in NHimp, NPimp and Restart

Peirce's rule is derivable in $NHimp$:

$$\dfrac{\vdash_i (A \to (A \to B)) \to (A \to B) \qquad [A]^1 \qquad \begin{array}{c}[A \to B]^2\\ \vdots \\ A\end{array}}{A}\;1,2$$

In fact, $NHimp$ is equivalent to $NPimp$ [82]. $NPimp$ consists of the usual intuitionistic rules for $\to$ plus Peirce's rule":

$$\dfrac{\begin{array}{c}[A \to B]^1\\ \vdots \\ A\end{array}}{A}\;1$$

Hosoi's rule is derivable in $NPimp$:

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\dfrac{[A \to B]^1 \qquad \begin{array}{c}\vdots\\ \Pi_1\\ (A \to B) \to B\end{array}}{B}\\ \Pi_3\\ C\end{array} \qquad [C \to A]^2}{A}}{A}\;1\\ \Pi_2\end{array}}{C}}{C}\;2$$

- Peirce's Rule is also derivable in NJ + restart:

$$\dfrac{\begin{array}{c}[A \to B]^*\\ \Pi_1\\ A\end{array}}{A*}\;\text{(Restart)*}$$

- Hosoi's Rule is also derivable in NJ + restart:

[82] For more on $NPimp$ viz. (PEREIRA et al., 2010) (ZIMMERMANN, 2002). Peirce's rule for Natural Deducion was proposed by Curry in (H.B. Curry, Foundations of Mathematical Logic, McGraw-Hill, New York, 1963)

$$\begin{matrix}\dfrac{\begin{matrix}\dfrac{\dfrac{\dfrac{[A]^1}{B}\,(\text{Restart})\ *}{A \to B}\,1 \qquad \begin{matrix}\Pi_1\\(A \to B) \to B\end{matrix}}{B}\\ \Pi_3\\ C\end{matrix} \qquad \dfrac{\dfrac{[C]^2}{A}\,(\text{Restart})\ *}{C \to A}\,2}{A^*}\\ \Pi_2\\ C^*\end{matrix}$$

The converse also holds:

- restart is derivable in NJ + Peirce [83]

$$\text{Peirce}\ \dfrac{\begin{matrix}\dfrac{A \qquad [A \to B]^1}{B}\\ \vdots\\ A\end{matrix}}{A}\,1$$

- restart is also derivable in NHimp:

$$\dfrac{\vdash_i (A \to (A \to B)) \to (A \to B) \qquad [A]^1 \qquad \begin{matrix}\dfrac{A \qquad [A \to B]^2}{B}\\ \vdots\\ A\end{matrix}}{A}\,1{,}2$$

In the next section, the notion of a normal form for $NHimp$ will be presented.

4.3.1.3 On $Himp$'s normal form

The notion of normal form is sensitive to the goal of the normalization strategy in question.

Prawitz's normalization procedure's [84] aim, for example, is to decrease the degree of the conclusions of the $\bot_c$ rule until all the conclusions of the $\bot_c$ rule are atomic. The procedure reduces the degree of these formulae until they are all atomic. A derivation is said to be normal if (1) no conclusion of an introduction rule is the major premise of an elimination rule and (2)

[83] this derivation was done in (GABBAY; GABBAY, 2005), pg.4, by Gabbay
[84] see (PRAWITZ, 1965)

all classical reasoning (i.e., applications of the $\perp_c$ rule) is restricted to the negation of atomic formulas.

Seldin's normalization procedure [85] has a different aim and, consequently, a different notion of normal form. The objective is to (i) reduce the number of applications of the $\perp_c$ rule in the derivation to at most one application and (ii) reduce the number of steps below the application of $\perp_c$ rule to zero. In other words, the aim is to reduce every classical proof Π to a proof Π^* where there is at most one application of $\perp_c$, and it is the last rule applied in Π^*. A derivation Π is said to be normal if (1) there is at most one application of $\perp_c$, in such a way that it is the last rule applied in Π and (2) no conclusion of an introduction rule is the major premise of an elimination rule.

Zimmerman's procedure [86] is similar to Prawitz's, but there is a little twist. Instead of working with the C system, Zimmerman's classical system consists of NJ + Peirce' rule. His aim is to restrict Peirce's rule to atomic conclusions.

Pereira, Costa, Haeusler, and Sanz's procedure is a combination of Zimmerman and Seldin. For the normalization of NHimp, we will use a strategy similar to (PEREIRA et al., 2010) Pereira, Costa, Hausler, and Sanz.

Quoting Guerrieri and Naibo: ”Our proof of the postponement of *reductio ad absurdum* [in our case, the postponement of Hosoi's rule] is proof-theoretic in a “geometric” way, in the sense that it relies on a notion of size for a derivation based only on the distance of the instances of *reductio ad abusrdum* from the conclusion of the derivation; the complexity of formulas plays no role in this definition of size.” (GUERRIERI; NAIBO, 2019, pg.3)

Definition 4.3.2 *Major Premise of the H-rule*

The Major premise of the H rule is : $((A \to B) \to B)$

Definition 4.3.3 $H - derivation$

A derivation Π is said to be a H-derivation if and only if it has the following form:

$$\dfrac{\vdash_i C \to (C \to P_i) \to (C \to P_i)) \qquad [C]^{ci} \qquad \begin{array}{c} \dfrac{\vdash_i C \to (C \to P_2) \to (C \to P_2)) \qquad [C]^{c2} \qquad \dfrac{\vdash_i C \to (C \to P_1) \to (C \to P_1)) \qquad [C]^{c1} \qquad \begin{array}{c}[C \to P_1]^1, [C \to P_2]^2 ..., [C \to P_i]^i] \\ \Pi_i \\ C\end{array}}{C}\, c1,1}{C}\, c2,2 \\ \vdots \\ C \end{array}}{C}\, ci, i$$

Where Π_i has no application of the H rule.

Definition 4.3.4 *Normal Derivation: A derivation Π is in normal form iff:*

---

[85] see (SELDIN, 1989)

[86] see (ZIMMERMANN, 2002)

- *(1) No major premise of an application of the $E \rightarrow$ Rule is the conclusion of an application of the $I \rightarrow$ Rule and there are no maximal segments.*

- *(2) The major premise of any application of the H Rule is an instance of the contraction schema: $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$*

- *(3) Π is an H-derivation*

4.3.1.4 Reductions

In this section, a series of reductions for the $NHimp$ system will be presented. The purpose of this procedure is to separate the applications of the H rule from the rest of the rules. The idea is that every proof Π of a formula A can be transformed into a proof Π' in such a way that Π' has two regions: an intuitionistic region Π_i and a classical region Π_c. Each reduction procedure for the H rule moves the rule "one step down", decreasing the number of non H rules below the H rule in question by 1. [87]

Definition 4.3.5 *Type (1) detour: the major premise of an application of the $E \rightarrow$ Rule is the conclusion of an application of the $I \rightarrow$ Rule.*

Definition 4.3.6 *$\rightarrow E$ reduction*

First, let's consider reductions for $\rightarrow I$ and $\rightarrow E$. They are the same as usual:
$\rightarrow E$ Reduction

$$\dfrac{\dfrac{\begin{matrix}[A]^n \\ \Pi_1 \\ B\end{matrix}}{A \rightarrow B}\, n \qquad \begin{matrix}\Pi_2 \\ A\end{matrix}}{B}$$

This derivation reduces to:

[87] The notation:

$$\begin{matrix}\Pi \\ [\alpha] \\ \Pi_1 \\ \beta\end{matrix}$$

denotes the substitution of the assumption class $[\alpha]$ by the derivation Π of α in Π_1

$$\begin{array}{c} \Pi_2 \\ [A] \\ \Pi_1 \\ B \end{array}$$

Definition 4.3.7 *Type (2) detours: The major premise of an application of the H Rule is not an instance of the contraction schema:* $(A \to (A \to B)) \to (A \to B)$[88]

Definition 4.3.8 $[HP]$ *reduction* [89]

Consider a derivation Π that contains the H rule:

$$\dfrac{\begin{array}{c} \Pi_1 \\ (A \to B) \to B \end{array} \quad \begin{array}{c} [A]^n \\ \Pi_2 \\ C \end{array} \quad \begin{array}{c} [B]^m \\ \Pi_3 \\ C \end{array}}{\begin{array}{c} C \\ \vdots \end{array}} \; n, m$$

$[HP]$ reduction procedure:[90][91]

[88] see definition 5.4

[89] The name HP reduction stands for Hosoi-Peirce reduction

[90] This procedure is useful because (i) when $[HP]$ is applied, the major premise of the H rule is transformed into the contraction schema $((\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta))$. Since the contraction schema can be proven intuitionistically, it follows that the derivation above the major premise of the H rule is guaranteed to be an intuitionistic derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\alpha]^1 \quad [\alpha \to (\alpha \to \beta)]}{\alpha \to \beta} \quad [\alpha]^1}{\beta}}{\alpha \to \beta} \, 1}{(\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta)}$$

[91] In the derivation below, one can see the relation between the H rule and Peirce's rule. First, we assume $[C \to B]$, we conclude C, and then, using the H rule, conclude C again, discharging the assumption of $[C \to B]$.

$$\dfrac{\vdash_i ((C \to (C \to B)) \to (C \to B) \qquad [C]^1 \qquad \begin{array}{c} \dfrac{\dfrac{\dfrac{\begin{array}{c}[A]^3\\ \Pi_2\\ C\end{array} \qquad [C \to B]^2}{B}}{A \to B}\,3 \qquad \begin{array}{c}\Pi_1\\ ((A \to B) \to B)\end{array}}{B} \\ \Pi_3 \\ C \end{array}}{\begin{array}{c} C \\ \vdots \end{array}}\,1,2$$

If Π_1 is intuitionistic and $(A \to B) \to B$ is obtained through $I \to$, then, we can apply the $E \to$ reduction:

$$\dfrac{\vdash_i (C \to (C \to B)) \to (C \to B) \qquad [C]^1 \qquad \begin{array}{c} \dfrac{\dfrac{\begin{array}{c}[A]^3\\ \Pi_2\\ C\end{array} \qquad [C \to B]^2}{B}}{A \to B}\,3 \\ \Pi_1 \\ B \\ \Pi_3 \\ C \end{array}}{\begin{array}{c} C \\ \vdots \end{array}}\,1,2$$

4.3.1.5 A small remark on $[HP]$ reductions

When implementing $[HP]$ reductions, one has to be careful. A simple change in the order of the minor premises of the H rule can result in a major premise that depends on classical reasoning:

$$\dfrac{\vdash_i ((C \to (C \to B)) \to (C \to B) \qquad [C]^1 \qquad \begin{array}{c} \dfrac{\dfrac{\dfrac{\begin{array}{c}[A]^3\\ \Pi_2\\ C\end{array} \qquad [C \to B]^2}{B}}{A \to B}\,3 \qquad \begin{array}{c}\Pi_1\\ (A \to B) \to B\end{array}}{B} \\ \Pi_3 \\ C \end{array}}{\begin{array}{c} C \\ \vdots \end{array}}\,1,2$$

if we change the positions of the minor premises, we then get Pierce's Law as the major premise.

$$\dfrac{\vdash_c (((C\to B)\to C)\to C) \qquad \begin{array}{c} \dfrac{\dfrac{\dfrac{\begin{array}{c}[A]^3\\ \Pi_2\\ C\end{array} \quad [C\to B]^2}{B}}{A\to B}\,3 \qquad \begin{array}{c}\Pi_1\\ (A\to B)\to B\end{array}}{B}\\ \Pi_3\\ C\end{array} \qquad [C]^1}{C}\,1,2$$

Since Pierce's Law is a classical theorem and not an intuitionistic one, its proof depends on classical reasoning. This means that if the major premise of an application of the H rule is a classically valid theorem, we would have to use the H rule in order to prove it. But that is not desirable if your objective is to permute the applications of the H rule with intuitionistic rules. That's why, when applying the $[HP]$ reduction procedure, one should always rely on the contraction schema as the major premise and not on Pierce's Law.

4.3.1.6 Permuting the H rule with $\to$

The aim of this subsection is to present procedures that eliminate type (3) detours.

Definition 4.3.9 *Type (3) detour: There is an application of an $I \to$ rule or of an $E \to$ rule below an application of an H rule.*

The permutation procedures for H and $\to$ function in a similar way to the procedures of (PEREIRA et al., 2010). The idea is to postpone the occurrences of the H rule, pilling them up at the end of the derivation. Since we are working with the implicational fragment [→], the number of applications of the H rule will not be reducible to one application.

There are several cases:

- (a) First case, permutation between $I \to$ and Hosoi's rule. The $I \to$ rule is being applied right after the $H - rule$.

$$\dfrac{\dfrac{\begin{array}{c}[P]_0\\ \Pi_1\\ ((A\to B)\to B)\end{array} \qquad \begin{array}{c}[A]_n[P]_0\\ \Pi_2\\ C\end{array} \qquad \begin{array}{c}[B]_m[P]_0\\ \Pi_3\\ C\end{array}}{C}\,n,m}{P\to C}\,0$$

First, apply the $[HP]$ reduction procedure:

$$
\dfrac{\vdash_i (C \to (C \to B)) \to (C \to B) \qquad [C]^3 \qquad \begin{matrix} \dfrac{\dfrac{\dfrac{\begin{matrix}[A]^2[P]^0 \\ \Pi_2 \\ C\end{matrix} \qquad [C \to B]^1}{B}}{A \to B}\,2 \qquad \begin{matrix}[P]_0 \\ \Pi_1 \\ ((A \to B) \to B)\end{matrix}}{[B]} \qquad \ldots \qquad [P]^0 \\ \Pi_3 \\ C \end{matrix}}{\dfrac{C}{P \to C}\,0}\,3,1
$$

Then, apply the following reduction procedure $[H_{I1}]$:

$$
\dfrac{\vdash_i (P \to C) \to ((P \to C) \to B) \to ((P \to C) \to B) \qquad [P \to C]^3 \qquad \begin{matrix} \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[A]^2 ... [P]^0 \\ \Pi_2 \\ C\end{matrix}}{P \to C} \qquad [(P \to C) \to B]^1}{B}}{A \to B}\,2 \qquad \begin{matrix}[P]^0 \\ \Pi_1 \\ (A \to B) \to B\end{matrix}}{[B]} \qquad \ldots \qquad [P]^0 \\ \Pi_3 \\ \dfrac{C}{P \to C}\,0 \end{matrix}}{P \to C}\,1,3
$$

We can also apply the following procedure $[H_{I2}]$ [92]:

$$
\dfrac{\vdash_i ((P \to C) \to ((P \to C) \to B)) \to ((P \to C) \to B) \qquad [(P \to C)]^2 \qquad \begin{matrix} \dfrac{\dfrac{\dfrac{\begin{matrix}[A]^3[P]^0 \\ \Pi_2 \\ C\end{matrix} \qquad \dfrac{\dfrac{\dfrac{[C]^4}{P \to C} \qquad [(P \to C) \to B]^1}{B}}{[C \to B]}\,4}{B}}{A \to B}\,3 \qquad \begin{matrix}[P]_0 \\ \Pi_1 \\ ((A \to B) \to B)\end{matrix}}{[B]} \qquad \ldots \qquad [P]_0 \\ \Pi_3 \\ \dfrac{C}{P \to C}\,0 \end{matrix}}{P \to C}\,1,2
$$

- (b) The second case is the permutation between $E \to$ and Hosoi's rule. The $E \to$ rule is being applied right after the $H - rule$. There are three subcases:

  (b.1) $[H_m]$ The minor premise is a conclusion of H.

  This reduction procedure permutes an application of the H-rule with an application of $\to -E$.

[92] in the procedure below, the maximum formula $[C \to B]$ is being introduced because it is easier in the $[H_{I2}]$ procedure to visualize that the hypothesis $C \to B$ that was being discharged in the original derivation is now being substituted by the hypothesis $(P \to C) \to B$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\((C\to D)\to D)\end{matrix}\qquad\begin{matrix}[C]^1\\\Pi_2\\A\end{matrix}\qquad\begin{matrix}[D]^2\\\Pi_3\\A\end{matrix}}{A}\,1,2\qquad\dfrac{\Pi_4}{A\to B}}{B}$$

First, apply the $[HP]$ reduction procedure:

$$\dfrac{\dfrac{\vdash_i (A\to(A\to D))\to(A\to D)\qquad [A]^2\qquad \begin{matrix}\dfrac{\dfrac{\dfrac{\begin{matrix}[C]^3\\\Pi_2\\A\end{matrix}\qquad [A\to D]^1}{D}}{C\to D}\,3\qquad\begin{matrix}\Pi_1\\((C\to D)\to D)\end{matrix}}{D}\\\Pi_3\\A\end{matrix}}{A}\,1,2\qquad\begin{matrix}\Pi_4\\A\to B\end{matrix}}{B}$$

Then, apply the following reduction procedure [Hm]:

$$\dfrac{\vdash_i ((B\to(B\to D))\to(B\to D)\qquad [B]^3\qquad \dfrac{\begin{matrix}\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]^2\qquad\dfrac{\Pi_4}{A\to B}}{B}\qquad [B\to D]^1}{D}}{A\to D}\,2\qquad\begin{matrix}[C]^4\\\Pi_2\\A\end{matrix}}{D}}{C\to D}\,4\qquad\begin{matrix}\Pi_1\\(C\to D)\to D\end{matrix}}{D}\\\Pi_3\\A\end{matrix}\qquad\begin{matrix}\Pi_4\\A\to B\end{matrix}}{B}}{B}\,3,1$$

(b.2) [H_M] The major premise is a conclusion of H

$$\dfrac{\begin{matrix}\Pi_1\\A\end{matrix}\qquad\dfrac{\begin{matrix}\Pi_2\\(C\to D)\to D\end{matrix}\qquad\begin{matrix}[C]^1\\\Pi_3\\A\to B\end{matrix}\qquad\begin{matrix}[D]^2\\\Pi_4\\A\to B\end{matrix}}{A\to B}\,1,2}{B}$$

First, apply the $[HP]$ reduction procedure:

$$\dfrac{\begin{matrix}\Pi_1\\A\end{matrix}\qquad\dfrac{\vdash_i ((A\to B)\to((A\to B)\to D))\to((A\to B)\to D)\qquad [A\to B]^2\qquad\begin{matrix}\dfrac{\dfrac{\dfrac{\begin{matrix}[C]^c\\\Pi_3\\A\to B\end{matrix}\qquad [(A\to B)\to D)]^1}{D}}{C\to D}\,c\qquad\begin{matrix}\Pi_2\\(C\to D)\to D\end{matrix}}{D}\\\Pi_4\\A\to B\end{matrix}}{A\to B}\,1,2}{B}$$

Then, apply the following reduction procedure $[H_M]$:

$$
\begin{prooftree}
\AxiomC{$\vdash_i (B \to (B \to D)) \to (B \to D)$}
\AxiomC{$[B]^4$}
\AxiomC{Π_1}\noLine\UnaryInfC{A}
\AxiomC{$[A \to B]^1$}
\BinaryInfC{B}
\AxiomC{$[B \to D]^2$}
\BinaryInfC{D}
\RightLabel{1}\UnaryInfC{$[(A \to B) \to D]$}
\AxiomC{$[C]^3$}\noLine\UnaryInfC{Π_3}\noLine\UnaryInfC{$A \to B$}
\BinaryInfC{D}
\RightLabel{3}\UnaryInfC{$C \to D$}
\AxiomC{Π_2}\noLine\UnaryInfC{$(C \to D) \to D$}
\BinaryInfC{D}
\noLine\UnaryInfC{Π_4}\noLine\UnaryInfC{$A \to B$}
\AxiomC{Π_1}\noLine\UnaryInfC{A}
\BinaryInfC{B}
\RightLabel{$4, 2$}\TrinaryInfC{B}
\end{prooftree}
$$

(b.3) Both the major premise and the minor premise are conclusions of the H rule. All the derivations Π are intuitionistic.

$$
\begin{prooftree}
\AxiomC{Π_{m1}}\noLine\UnaryInfC{$((C \to D) \to D)$}
\AxiomC{$[C]^1$}\noLine\UnaryInfC{Π_1}\noLine\UnaryInfC{A}
\AxiomC{$[D]^2$}\noLine\UnaryInfC{Π_2}\noLine\UnaryInfC{A}
\RightLabel{$1, 2$}\TrinaryInfC{A}
\AxiomC{Π_{m2}}\noLine\UnaryInfC{$(E \to F) \to F$}
\AxiomC{$[E]^3$}\noLine\UnaryInfC{Π_3}\noLine\UnaryInfC{$A \to B$}
\AxiomC{$[F]^4$}\noLine\UnaryInfC{Π_4}\noLine\UnaryInfC{$A \to B$}
\RightLabel{$3, 4$}\TrinaryInfC{$A \to B$}
\BinaryInfC{B}
\end{prooftree}
$$

First, apply the $[H_p]$ reduction procedure:

$$
\begin{prooftree}
\AxiomC{$\vdash_i (A \to (A \to D)) \to (A \to D)$}
\AxiomC{$[A]^1$}
\AxiomC{$[C]^2$}\noLine\UnaryInfC{Π_1}\noLine\UnaryInfC{A}
\AxiomC{$[A \to D]^3$}
\BinaryInfC{D}
\RightLabel{2}\UnaryInfC{$C \to D$}
\AxiomC{Π_{m1}}\noLine\UnaryInfC{$(C \to D) \to D$}
\BinaryInfC{D}
\noLine\UnaryInfC{Π_2}\noLine\UnaryInfC{A}
\RightLabel{$1, 3$}\TrinaryInfC{A}
\AxiomC{$\vdash_i ((A \to B) \to ((A \to B) \to F)) \to ((A \to B) \to F)$}
\AxiomC{$[A \to B]^6$}
\AxiomC{$[E]^5$}\noLine\UnaryInfC{Π_3}\noLine\UnaryInfC{$A \to B$}
\AxiomC{$[((A \to B) \to F)]^4$}
\BinaryInfC{F}
\RightLabel{5}\UnaryInfC{$E \to F$}
\AxiomC{Π_{m2}}\noLine\UnaryInfC{$(E \to F) \to F$}
\BinaryInfC{F}
\noLine\UnaryInfC{Π_4}\noLine\UnaryInfC{$A \to B$}
\RightLabel{$4, 6$}\TrinaryInfC{$A \to B$}
\BinaryInfC{B}
\end{prooftree}
$$

Then, apply the following reduction procedure $[H_{Mm}]$:

$$
\begin{prooftree}
\AxiomC{$\vdash_i ((B \to (B \to D)) \to (B \to D)$}
\AxiomC{$[B]^{d2}$}
\AxiomC{$\vdash_i ((B \to (B \to F)) \to (B \to F)$}
\AxiomC{$[B]^{d1}$}
\AxiomC{$\vdash_i (A \to (A \to D)) \to (A \to D)$}
\AxiomC{$[A]^8$}
\AxiomC{$[C]^9$}\noLine\UnaryInfC{Π_1}\noLine\UnaryInfC{A}
\AxiomC{$[A \to D]^7$}
\BinaryInfC{D}
\RightLabel{9}\UnaryInfC{$C \to D$}
\AxiomC{Π_{m1}}\noLine\UnaryInfC{$(C \to D) \to D$}
\BinaryInfC{D}
\noLine\UnaryInfC{Π_2}\noLine\UnaryInfC{A}
\RightLabel{$8, 7$}\TrinaryInfC{A}
\AxiomC{$[A]^1$}
\AxiomC{$[A \to B]^2$}
\BinaryInfC{B}
\AxiomC{$[B \to D]^3$}
\BinaryInfC{D}
\RightLabel{1}\UnaryInfC{$A \to D$}
\AxiomC{$[C]^4$}\noLine\UnaryInfC{Π_1}\noLine\UnaryInfC{A}
\BinaryInfC{D}
\RightLabel{4}\UnaryInfC{$C \to D$}
\AxiomC{Π_{m1}}\noLine\UnaryInfC{$(C \to D) \to D$}
\BinaryInfC{D}
\noLine\UnaryInfC{Π_2}\noLine\UnaryInfC{A}
\AxiomC{$[A \to B]^2$}
\BinaryInfC{B}
\AxiomC{$[B \to F]^5$}
\BinaryInfC{F}
\RightLabel{2}\UnaryInfC{$[((A \to B) \to F)]$}
\AxiomC{$[E]^6$}\noLine\UnaryInfC{Π_3}\noLine\UnaryInfC{$A \to B$}
\BinaryInfC{F}
\RightLabel{6}\UnaryInfC{$E \to F$}
\AxiomC{Π_{m2}}\noLine\UnaryInfC{$(E \to F) \to F$}
\BinaryInfC{F}
\noLine\UnaryInfC{Π_4}\noLine\UnaryInfC{$A \to B$}
\BinaryInfC{B}
\RightLabel{$d1, 5$}\TrinaryInfC{B}
\RightLabel{$d2, 3$}\TrinaryInfC{B}
\end{prooftree}
$$

Now, consider a derivation Π where (i) the last rule applied is the $\to E$ rule, (ii) the derivation of the minor premise is a H-Derivation and (iii) the derivation of the major premise is the result of an application of the H rule:

$$
\begin{prooftree}
\AxiomC{$\vdash_i (A \to (A \to P_n)) \to (A \to P_n)$}
\AxiomC{$[A]^{an}$}
\AxiomC{$\vdash_i (A \to (A \to P_2)) \to (A \to P_2)$}
\AxiomC{$[A]^{a2}$}
\AxiomC{$\vdash_i (A \to (A \to P_1)) \to (A \to P_1))$}
\AxiomC{$[A]^{a1}$}
\AxiomC{$[A \to P_1]^{p1}, [A \to P_2]^{p2} ..., [A \to P_n]^{pn}$}\noLine\UnaryInfC{$\Pi_1$}\noLine\UnaryInfC{$A$}
\RightLabel{$a1, p1$}\TrinaryInfC{A}
\RightLabel{$a2, p2$}\TrinaryInfC{A}
\noLine\UnaryInfC{$\vdots$}\noLine\UnaryInfC{A}
\RightLabel{an, pn}\TrinaryInfC{A}
\AxiomC{$\vdash_i ((A \to B) \to ((A \to B) \to Q)) \to ((A \to B) \to Q)$}
\AxiomC{$[(A \to B)]^b$}
\AxiomC{$[(A \to B) \to Q]^q$}\noLine\UnaryInfC{Π_2}\noLine\UnaryInfC{$A \to B$}
\RightLabel{b,q}\TrinaryInfC{$A \to B$}
\BinaryInfC{B}
\end{prooftree}
$$

The derivation above reduces to

$[H_{Mm*}]$ reduction procedure:

$$\dfrac{\vdash_i (B \to (B \to Q)) \to (B \to Q) \quad [B]^{b(n+1)} \quad \dfrac{\vdash_i (B \to (B \to P_n)) \to (B \to P_n) \quad [B]^{bn} \quad \begin{matrix} \dfrac{\vdash_i (B \to (B \to P_1)) \to (B \to P_1) \quad [B]^{b_1} \quad \dfrac{\begin{matrix} \Pi^* \\ A \end{matrix} \quad \begin{matrix} \dfrac{\dfrac{[B \to Q]^q \quad \dfrac{[A \to B]^2 \quad \begin{matrix} \dfrac{\dfrac{\dfrac{[A]^1 \quad [A \to B]^2}{B} \quad [B \to P_1]^{p_1}}{P_1}}{[A \to P_1]}\,1 \quad \dots \quad \dfrac{\dfrac{\dfrac{A \quad [A \to B]^2}{B} \quad [B \to P_n]^{p_n}}{P_n}}{[A \to P_n]} \\ \Pi_1 \\ A \end{matrix}}{[B]}}{Q}}{[(A \to B) \to Q]}\,2 \\ \Pi_2 \\ A \to B \end{matrix}}{B}}{B}\,p_1, b_1 \\ \vdots \\ B \end{matrix}}{B}\,p_n, b_n}{B}\,q, b(n+1)$$

Where Π^* is:

$$\dfrac{\vdash_i (A \to (A \to P_i)) \to (A \to P_i) \quad [A]^{ai} \quad \begin{matrix} \dfrac{\vdash_i (A \to (A \to P_2)) \to (A \to P_2) \quad [A]^{a2} \quad \dfrac{\vdash_i (A \to (A \to P_1)) \to (A \to P_1) \quad [A]^{a1} \quad \begin{matrix} [A \to P_1]^{p1}, [A \to P_2]^{p2} ..., [A \to P_i]^{pi} \\ \Pi_n \\ A \end{matrix}}{A}\,a1, p1}{A}\,a2, p2 \\ \vdots \\ A \end{matrix}}{A}\,ai, pi$$

4.3.1.7 Normalization of $NHimp$

Definition 4.3.10 *Critical derivation*

A derivation Π is called a critical derivation iff it satisfies the following conditions:

- *The last rule applied in Π is not an application of the H-Rule*
- *The derivation(s) of the premise(s) of the last rule applied in Π is a (are) $H-derivation(s)$.*

Definition 4.3.11 *Intuitionistic part: Given a derivation Π, the intuitionistic parts of Π are a sequence of steps in Π that contains no applications of the $H - rule$*

Definition 4.3.12 *Classical part: Given a derivation Π, the classical parts of Π are sequences of steps of Π that contains only applications of the $H - rule$*

Definition 4.3.13 *H-Normal Form (HNF)*

A H-derivation is said to be in H-normal-form (HNF) iff its intuitionistic part is normal. No formula in the intuitionistic part is at the same time the conclusion of $\to -I$ rule and major premisse of $\to -E$ rule.

Lemma 4.3.14 (Critical Lemma) *Let Π be a critical derivation of A from Γ in NHimp. Then Π can be tranformed into a H-derivation Π' of A from Γ.*

The proof is by induction on the number of applications of the H-rule in Π

Basis In case Π does not contain any application of the H-rule, then we take $\Pi' = \Pi$

Inductive Step

- **1.** The last rule applied in Π is $I \to$ and Π has k applications of the H-rule. Π has the following form:

$$\dfrac{\dfrac{C \to (C\to P_k)\to(C\to P_k)) \quad [C]^{ci} \quad \begin{array}{c} \dfrac{C\to(C\to P_2)\to(C\to P_2)) \quad [C]^{c2} \quad \dfrac{C\to(C\to P_1)\to(C\to P_1)) \quad [C]^{c1} \quad \begin{array}{c}[C\to P_1]^{p1}, [C\to P_2]^{p2}..., [C\to P_k]^{pk}...[Q]^q\\ \Pi' \\ C\end{array}}{C}\,p1,c1}{C}\,p2,c2 \\ \vdots \\ C\end{array}}{C}\,pk,ci}{Q\to C}\,q$$

Where Π' is an intuitionistic derivation

By an application of $[H_{I2}]$, Π reduces to:

$$\dfrac{\vdash_i ((Q\to C)\to((Q\to C)\to P_k))\to((Q\to C)\to P_k) \quad [Q\to C]^{q\to c} \quad \dfrac{\dfrac{\vdash_i C\to(C\to P_{k-1})\to(C\to P_{k-1})) \quad [C]^{ci} \quad \begin{array}{c}\dfrac{\vdash_i C\to(C\to P_1)\to(C\to P_1)) \quad [C]^{c1} \quad \begin{array}{c}[Q]^q \quad [C\to P_1]^{p1} \quad ... \quad \dfrac{\dfrac{[C]^*}{Q\to C} \quad [(Q\to C)\to P_k]^{pk}}{\dfrac{P_k}{[C\to P_k]}\,*} \\ \Pi' \\ C\end{array}}{C}\,p1,c1 \\ \vdots \\ C\end{array}}{C}\,ci,pk-1}{Q\to C}\,q}{Q\to C}\,pk,q\to c$$

The result now follows directly from the induction hypothesis applied to the derivation of the premise of the last application of the H rule in the derivation above.

If we repeat this process k times, we get:

$$\dfrac{Q\to C\to(Q\to C\to P_k)\to(C\to P_k)) \quad [Q\to C]^{ci} \quad \begin{array}{c}\dfrac{C\to(C\to P_1)\to(C\to P_1)) \quad [Q\to C]^{c1} \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{[C]}{Q\to C} \quad [(Q\to C)\to P_1]^1}{\dfrac{P_1}{[C\to P_1]}} \quad \dots \quad \dfrac{\dfrac{[C]}{Q\to C} \quad [(Q\to C)\to P_k]^k}{\dfrac{P_k}{[C\to P_k]}} \quad [Q]^q \\ \Pi_n \\ C\end{array}}{Q\to C}\,q}{Q\to C}\,c1,1 \\ \vdots \\ Q\to C\end{array}}{Q\to C}\,ci,k$$

- **2.** The last rule applied to Π is the $\to -E$ rule. There are three sub cases, (a) When Π is a critical derivation ending in the $E \to$ rule, where the major premise of the $E \to$ rule is an $H - derivation$ (b) When Π is a critical derivation ending in the $E \to$ rule, where the minor premise of the $E \to$ rule is an $H - derivation$ (c) When Π is a critical derivation ending in the $E \to$ rule, where both the minor and the major premises of the $E \to$ rule are $H - derivations$:

(a) Π is:

$$
\dfrac{\dfrac{\vdash_i ((A \to B) \to ((A \to B) \to Q) \to ((A \to B) \to Q) \qquad [A \to B]^{a \to b} \qquad \begin{array}{c} \dfrac{\vdash_i ((A \to B) \to ((A \to B) \to Q))) \to ((A \to B) \to Q) \qquad [A \to B] \qquad \begin{array}{c}[(A \to B) \to Q_1]^{q1}...[(A \to B) \to Q_n]^{qn} \\ \Pi_2' \\ A \to B\end{array}}{A \to B}\,{\scriptstyle q1, a \to b} \\ \vdots \\ A \to B \end{array}}{A \to B}\,{\scriptstyle qn, a \to b} \qquad \begin{array}{c}\Pi_1' \\ A\end{array}}{B}
$$

Where Π_1' and Π_2' are intuitionistic derivations.

By the $[H_M]$ reduction we get:

$$
\dfrac{(B \to (B \to Q_n)) \to (B \to Q_n) \qquad [B] \qquad \dfrac{\dfrac{\Pi_1}{A} \qquad \dfrac{((A \to B) \to (A \to B) \to Q_{n-1})) \to ((A \to B) \to Q_{n-1}) \qquad [(A \to B)]^{a \to b} \qquad \begin{array}{c} \dfrac{((A \to B) \to (A \to B) \to Q_1)) \to ((A \to B) \to Q_1) \qquad [(A \to B)]^{a \to b} \qquad \begin{array}{c} \dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1 \\ A\end{array} \qquad [A \to B]^{a \to b}}{B} \qquad [B \to Q_n]^{qn}}{Q_n}}{[(A \to B) \to Q_n]}\,{\scriptstyle a \to b} \quad ... \quad [(A \to B) \to Q_1]^{q1} \\ \Pi_2' \\ A \to B \end{array}}{A \to B}\,{\scriptstyle a \to b, q1} \\ \vdots \\ A \to B \end{array}}{A \to B}\,{\scriptstyle b, qn-1}}{B}}{B}\,{\scriptstyle qn}
$$

The result now follows directly from the induction hypothesis applied to the derivation of the premise of the last application of the H rule in the derivation above. If we repeat this process n times, we get:

$$
\dfrac{(B \to (B \to Q_n)) \to (B \to Q_n) \qquad [B]^b \qquad \begin{array}{c} \dfrac{(B \to (B \to Q_1)) \to (B \to Q_1) \qquad [B]^b \qquad \dfrac{\begin{array}{c} \dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1' \\ A\end{array} \qquad [A \to B]^{a \to b}}{B} \qquad [B \to Q_1]^{q1}}{Q_1}}{[(A \to B) \to Q_1]}\,{\scriptstyle a \to b} \quad ... \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1' \\ A\end{array} \qquad [A \to B]^{a \to b}}{B} \qquad [B \to Q_n]^{q_n}}{Q_n}}{[(A \to B) \to Q_n]}\,{\scriptstyle a \to b} \\ \Pi_2' \\ A \to B \end{array} \qquad \begin{array}{c}\Pi_1' \\ A\end{array}}{B}}{B}\,{\scriptstyle b, q1} \\ \vdots \\ B \end{array}}{B}\,{\scriptstyle b, qn}
$$

(b) Π is:

$$
\dfrac{\dfrac{\vdash_i (A \to (A \to Q_n)) \to (A \to Q_n) \qquad [A]^a \qquad \begin{array}{c} \dfrac{\vdash_i (A \to (A \to Q_1)) \to (A \to Q_1) \qquad [A]^a \qquad \begin{array}{c}[A \to Q_1]^{q1}, ..., [A \to Q_n]^{qn} \\ \Pi_1' \\ A\end{array}}{A}\,{\scriptstyle q1, a} \\ \vdots \\ A \end{array}}{A}\,{\scriptstyle qn, a} \qquad \begin{array}{c}\Pi_2' \\ A \to B\end{array}}{B}
$$

Where Π_1' and Π_2' are intuitionistic derivations.

By the $[H_m]$ reduction, we get:

$$
\dfrac{\vdash_i (B \to (B \to Q_n)) \to (B \to Q_n) \qquad [B]^b \qquad \dfrac{\dfrac{\vdash_i (A \to (A \to Q_{(n-1)})) \to (A \to Q_{(n-1)}) \qquad [A]^a \qquad \begin{array}{c} \dfrac{\vdash_i (A \to (A \to Q_1)) \to (A \to Q_1) \qquad [A]^a \qquad \begin{array}{c} [A \to Q_1]^{q1} \quad ... \quad \dfrac{\dfrac{\dfrac{[A]^a \qquad \begin{array}{c}\Pi_4 \\ A \to B\end{array}}{B} \qquad [B \to Q_n]^{q_n}}{Q_n}}{[A \to Q_n]}\,{\scriptstyle a} \\ \Pi_1' \\ A \end{array}}{A}\,{\scriptstyle q1, a} \\ \vdots \\ A \end{array}}{A}\,{\scriptstyle qn-1, a} \qquad \begin{array}{c}\Pi_2 \\ A \to B\end{array}}{B}}{B}\,{\scriptstyle qn, b}
$$

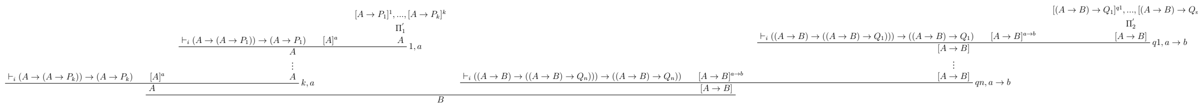
The result now follows directly from the induction hypothesis applied to the derivation of the premise of the last application of the H rule in the derivation above.

If we apply $[H_m]$ n times we get:

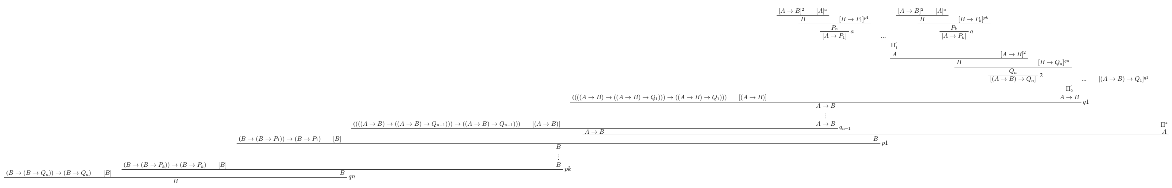
$$
\dfrac{\vdash_i (B \to (B \to Q_n)) \to (B \to Q_n) \qquad [B]^b \qquad \begin{array}{c} \dfrac{\vdash_i (B \to (B \to Q_1)) \to (B \to Q_1) \qquad [B]^b \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[A]^a \quad \begin{array}{c}\Pi_2\\ A \to B\end{array}}{B} \quad [B \to Q_1]^{q1}}{Q_1}}{[A \to Q_1]}\,a \quad \ldots \quad \dfrac{\dfrac{\dfrac{[A]^a \quad \begin{array}{c}\Pi_2\\ A \to B\end{array}}{B} \quad [B \to Q_n]^{qn}}{Q_1}}{[A \to Q_n]}\,a \\ \begin{array}{c}\Pi_1\\ A\end{array} \qquad \begin{array}{c}\Pi_2\\ A \to B\end{array}\end{array}}{B}\,\text{q1,b}}{B} \\ \vdots \\ B \end{array}}{B}\,\text{qn,b}
$$

(c) Both the minor and the major premises of the $E \to$ rule are $H-$derivations.

In this case, Π is as it follows:



Where Π_1' and Π_2' are intuitionistic parts of Π. By an application of the $[H_{Mm*}]$ procedure we get:



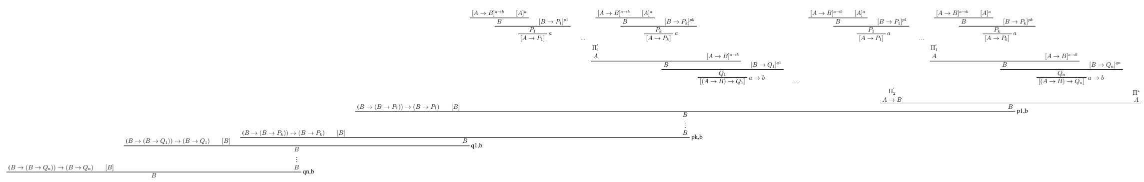
Where Π^* is:

$$
\dfrac{(A \to (A \to P_k)) \to (A \to P_k) \qquad [A]^{ai} \qquad \begin{array}{c} \dfrac{(A \to (A \to P_2)) \to (A \to P_2) \qquad [A]^{a2} \qquad \dfrac{(A \to (A \to P_1)) \to (A \to P_1) \qquad [A]^{a1} \qquad \begin{array}{c}[A \to P_1]^{p1}, [A \to P_2]^{p2} ..., [A \to P_k]^{pk}\\ \Pi_n \\ A\end{array}}{A}\,a1, p1}{A}\,a2, p2 \\ \vdots \\ A \end{array}}{A}\,ai, pk
$$

The result now follows directly from the induction hypothesis applied to the derivation of the premise of the last application of the H rule in the derivation above.

Note that each time we apply this process, the premisses $[A \to P_1]...[A \to P_k]$ will be doubled!

If we apply $[H_{Mm*}]$ n times we get:



Where Π^* continues the same.

Below, a simplified version:

$$
\begin{array}{c}
\Delta_1 \quad \dots \quad \Delta_k \\
\Pi_1' \\
\dfrac{\dfrac{\dfrac{A \qquad [A \to B]^{a\to b}}{B} \qquad [B \to Q_1]^{q1}}{Q_1}}{[(A \to B) \to Q_1]}\, a \to b
\end{array}
\quad \dots \quad
\begin{array}{c}
\Delta_1 \quad \dots \quad \Delta_k \\
\Pi_1' \\
\dfrac{\dfrac{\dfrac{A \qquad [A \to B]^{a\to b}}{B} \qquad [B \to Q_n]^{qn}}{Q_n}}{[(A \to B) \to Q_n]}\, a \to b \\
\dfrac{\begin{array}{c}\Pi_2' \\ A \to B\end{array} \qquad\qquad \begin{array}{c}\Pi^* \\ A\end{array}}{\text{H rule}\ \dfrac{B}{B}\ \text{p1,b}} \\
\vdots \\
\text{H rule}\ \dfrac{B}{B}\ \text{pk,b} \\
\text{H rule}\ \dfrac{B}{B}\ \text{q1,b} \\
\vdots \\
\text{H rule}\ \dfrac{B}{B}\ \text{qn,b}
\end{array}
$$

Where the applications of the H rule have been omitted and Δ_i stands for:

$$
\dfrac{\dfrac{\dfrac{[A \to B]^{a\to b} \qquad [A]^a}{B} \qquad [B \to P_i]^{pi}}{P_1}}{[A \to P_i]}\, a
$$

Having defined the reduction procedures above, it can be seen that the normalization results for (PEREIRA et al., 2010, pg.103-104) carry over:

Theorem 4.3.15 *Theorem (Normalization). If* Π *is a derivation of* A *from* Γ *in NHimp, then* Π *reduces to a HNF derivation* Π^* *of* A *from* Γ.

The proof proceeds by induction on the length of Π.

Basis: the base case is trivial.

Inductive step

1. The last rule applied in Π is not an application of the H-rule. By the induction hypothesis, the derivations of each premise can be reduced to H derivations in normal form. By the critical lemma, we obtain a H-derivation Π^* of A from Γ. A normal derivation Π^* of A from Γ can be obtained by the usual normalization technique applied to the intuitionistic part of Π^*.

2. The last rule applied in Π is an application of a H-rule. The result follows directly from the induction hypothesis.

4.3.1.8 NH_{imp} and the $<\to>$ fragment of classical logic

NH_{imp} is a Strictly Analytical system: It has the separation property, but it does not have the subformula property [93]. Here is a derivation of Peirce's axiom:

[93] Although the separation property is not interesting here. There is only one connective

$$\dfrac{\vdash_i (A \to (A \to B)) \to (A \to B) \qquad [A]^1 \qquad \dfrac{[A \to B]^2 \qquad [(A \to B) \to A]^3}{A}}{\dfrac{A}{((A \to B) \to A) \to A}\, 3}\, 1,2$$

Peirce's axiom is proven using only implicational axioms. As it was shown, the separation property holds. But, the derivation above clearly breaks the subformula property: the formula $(A \to (A \to B)) \to (A \to B)$, the major premiss of the application of the H rule, isn't a subformula of the conclusion $((A \to B) \to A) \to A$. The NH_{imp} is interesting because it can prove the whole implicational fragment of classical logic using only implicational rules.

4.3.2 [$\to, \neg$] fragment - $NHimp_\perp$ system]

The addition of $\perp$ to $NHimp$ allows one to define negation in the system: $\neg A$ is defined as $A \to \perp$. Let's call it $NHimp_\perp$ (Where N stands for negation). The addition of $\perp$ comes with a new rule:

- $ExFalsoQuodLibet$

$$\frac{\perp}{A}$$

This small change results in a complete system in relation to propositional classical logic. As it is well known, the fragment of Prawitz's C system [$\to, \perp$] is complete. It is strong enough to define all logical constants and to prove all classical theorems. The $NHimp_\perp$ system is also complete:

Definition 4.3.16 *Table of definitions for $NHimp\perp$*

- *Negation*

  $\neg A := A \to \perp$

- *Disjunction*

  $A \vee B := (A \to B) \to B$

- *Conjunction*

  $A \wedge B := (A \to (B \to \perp)) \to \perp$

All the rules for the operators above are derivable in $NHimp\perp$:

- The translation of negation rules is straightforward. It is the usual constructive interpretation of negation.

  $\neg I$

$$\dfrac{\begin{array}{c}[A]^1\\ \vdots\\ \bot\end{array}}{\neg A}\,1$$

  Translates to:

$$\dfrac{\begin{array}{c}[A]^1\\ \vdots\\ \bot\end{array}}{A \to \bot}\,1$$

  ¬ E

$$\dfrac{A \qquad \neg A}{\bot}$$

  translates to:

$$\dfrac{A \qquad A \to \bot}{\bot}$$

- Disjunction

  The same as for $NHimp$

- Conjunction

  $A \wedge B := (A \to (B \to \bot)) \to \bot$

  $I\wedge$

$$\dfrac{\dfrac{\dfrac{[A \to (B \to \bot)]^1 \qquad \begin{array}{c}\vdots\\ A\end{array}}{B \to \bot} \qquad \begin{array}{c}\vdots\\ B\end{array}}{\bot}}{((A \to (B \to \bot)) \to \bot}\,1$$

$E\wedge$ left

$$\dfrac{\vdash (A\to(A\to(B\to\bot)))\to(A\to(B\to\bot)) \qquad [A]^1 \qquad \dfrac{\dfrac{[A\to(B\to\bot)]^2 \qquad \begin{matrix}\vdots\\ (A\to(B\to\bot))\to\bot\end{matrix}}{\bot}}{A}}{A}\,1,2$$

$E\wedge$ right

$$\dfrac{\vdash (B\to(B\to\bot))\to(B\to\bot) \qquad [B]^1 \qquad \dfrac{\dfrac{\dfrac{[(B\to\bot)]^2}{A\to(B\to\bot)} \qquad \begin{matrix}\vdots\\ (A\to(B\to\bot))\to\bot\end{matrix}}{\bot}}{B}}{B}\,1,2$$

4.3.2.1 $\perp$ reduction

Consider the following derivation Π:

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ (A\to B)\to B\end{matrix} \qquad \begin{matrix}[A]^1\\ \Pi_2\\ \bot\end{matrix} \qquad \begin{matrix}[B]^2\\ \Pi_3\\ \bot\end{matrix}}{\bot}\,1,2}{C}$$

First, let's apply the $[HP]$ reduction procedure:

$$\dfrac{\dfrac{((\bot\to(\bot\to C))\to(\bot\to C) \qquad [\bot]^3 \qquad \begin{matrix}\dfrac{\dfrac{\dfrac{\begin{matrix}[A]^1\\ \Pi_2\\ \bot\end{matrix} \qquad [\bot\to B]^2}{B}}{A\to B}\,1 \qquad \begin{matrix}\Pi_1\\ (A\to B)\to B\end{matrix}}{[B]}\\ \Pi_3\\ \bot\end{matrix}}{\bot}\,2,3}{C}$$

the derivation above reduces to:

$[H\perp]$ reduction procedure

$$
\dfrac{\dfrac{\begin{matrix}[A]^1\\ \Pi_2\\ \bot\end{matrix} \qquad \dfrac{\dfrac{[\bot]^2}{B}}{[\bot \to B]}\,2}{B}}{A \to B}\,1 \qquad \begin{matrix}\Pi_1\\ (A \to B) \to B\end{matrix}
$$
$$
\begin{matrix}[B]\\ \Pi_3\\ \dfrac{\bot}{C}\end{matrix}
$$

Now, consider the following derivation critical derivation Π:

$$
\dfrac{\bot \to (\bot \to P_k) \to (\bot \to P_k)) \qquad [\bot]^{\bot_i} \qquad \begin{matrix}\dfrac{\bot \to (\bot \to P_1) \to (\bot \to P_1)) \qquad [\bot]^{\bot_1} \qquad \begin{matrix}[\bot \to P_1]^{p1}, ..., [\bot \to P_k]^{pk}\\ \Pi_i\\ \bot\end{matrix}}{\bot}\,\bot_1, p1\\ \vdots\\ \bot\end{matrix}}{\dfrac{\bot}{B}}\,\bot_i, pk
$$

If we apply the $[H\bot]$ reduction procedure we get:

$[H_\bot]$ reduction

$$
\dfrac{\bot \to (\bot \to P_{k-1}) \to (\bot \to P_{k-1})) \qquad [\bot]^{\bot_i} \qquad \begin{matrix}\dfrac{\bot \to (\bot \to P_1) \to (\bot \to P_1)) \qquad [\bot]^{\bot_1} \qquad \begin{matrix}\dfrac{\dfrac{[\bot]^{\bot}}{P_k}}{\bot \to P_k}\,\bot \quad [\bot \to P_1]^{p1} \quad , ... , \quad [\bot \to P_k - 1]^{pk-1}\\ \Pi_i\\ \bot\end{matrix}}{\bot}\,\bot_1, p1\\ \vdots\\ \bot\end{matrix}}{\dfrac{\bot}{B}}\,\bot_i, pk-1
$$

Each application of $[H_\bot]$ reduction reduces by 1 the number of applications of the H rule. The derivation above can be reduced to:

$$
\begin{matrix}\dfrac{\dfrac{[\bot]}{P_1}}{\bot \to P_1} \qquad ... \qquad \dfrac{\dfrac{[\bot]}{P_k}}{\bot \to P_k}\\ \Pi'\\ \dfrac{\bot}{B}\end{matrix}
$$

4.3.2.2 $NHimp_\bot$ useful theorems

1. LEM: $(A \vee \neg A)$

$$\dfrac{\vdash_i (A \to (A \to \bot)) \to (A \to \bot) \qquad \dfrac{[A]^1}{(A \vee \neg A)} \qquad \dfrac{[(A \to \bot)]^2}{(A \vee \neg A)}}{(A \vee \neg A)}\ 1, 2$$

2. Reduction ad absurdum $(\neg A \to \bot) \vdash A$

$$\dfrac{\vdash_i (A \to (A \to \bot)) \to (A \to \bot) \qquad [A]^1 \qquad \dfrac{\dfrac{[\neg A]^2 \qquad [\neg A \to \bot]}{\bot}}{A}\ \text{ExFalso}}{A}\ 1, 2$$

3. DNE - Double negation elimination $((A \to \bot) \to \bot) \to A$

$$\dfrac{\dfrac{[((A \to \bot) \to \bot)]^1 \qquad [A]^2 \qquad \dfrac{[\bot]^3}{A}}{A}\ 2, 3}{((A \to \bot) \to \bot) \to A}\ 1$$

4. Consequentia Mirabilis

$$\dfrac{\dfrac{\vdash_i (A \to (A \to \bot)) \to (A \to \bot) \qquad [A]^1 \qquad \dfrac{[A \to \bot]^2 \qquad [(A \to \bot) \to A]^3}{A}}{A}\ 1, 2}{((A \to \bot) \to A) \to A}\ 3$$

4.4 NH

Let's define NH as the system composed of Hosoi's rule plus the propositional fragment of Gentzen's NJ system.

4.4.1 NH definitions

Definition 4.4.1 $GH - derivation$[94]

A derivation Π is said to be a GH-derivation if and only if it has the following form:

$$\dfrac{\vdash_i C \to (C \to P_i) \to (C \to P_i)) \qquad \begin{array}{c}[C]^{ci}\\ \Pi_n\\ A\end{array} \qquad \begin{array}{c}\dfrac{\vdash_i C \to (C \to P_2) \to (C \to P_2)) \qquad \begin{array}{c}[C]^{c2}\\ \Pi_3\\ A\end{array} \qquad \dfrac{\vdash_i C \to (C \to P_1) \to (C \to P_1)) \qquad \begin{array}{c}[C]^{c1}\\ \Pi_2\\ A\end{array} \qquad \begin{array}{c}[C \to P_1]^1, [C \to P_2]^2 ..., [C \to P_i]^i\\ \Pi_1\\ A\end{array}}{A}\, c1,1}{A}\, c2,2\\ \vdots\\ A\end{array}}{A}\, ci,i$$

Where Π_i has no application of the H rule.

Definition 4.4.2 *Normal Derivation: A derivation Π is in normal form iff:*

- *(1) No major premise of an application of an $Elimination$ Rule is the conclusion of an application of an $Introduction$ Rule and there are no maximal segments.*

- *(2) The major premise of every application of the H Rule is an instance of the contraction schema: $(A \to (A \to B)) \to (A \to B)$*

- *(3) Π is a GH-derivation*

4.4.1.1 $\wedge$ permutations

- $[HP_{E\wedge}]$

  Consider the following derivation:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ (C \to D) \to D\end{array} \qquad \begin{array}{c}[C]^1\\ \Pi_2\\ A \wedge B\end{array} \qquad \begin{array}{c}[D]^2\\ \Pi_3\\ A \wedge B\end{array}}{A \wedge B}\, 1,2}{A}$$

  First, apply the [HP] reduction procedure:

[94] because of the $I\vee$ and $E\vee$ permutation procedures, a more general notion of $H - derivation$ will be used. Let's call it $GH - derivation$ (where GH stands for General Hosoi)

$$
\dfrac{\dfrac{\vdash_i ((A \wedge B) \to ((A \wedge B) \to D)) \to ((A \wedge B) \to D) \qquad [A \wedge B]^1 \qquad \begin{array}{c} \dfrac{\dfrac{\dfrac{\begin{array}{c}[C]^3 \\ \Pi_2 \\ A \wedge B\end{array} \quad [(A \wedge B) \to D]^2}{D}}{C \to D}\,3 \qquad \begin{array}{c}\Pi_1 \\ (C \to D) \to D\end{array}}{[D]} \\ \Pi_3 \\ A \wedge B \end{array}}{A \wedge B}\,1,2}{A}
$$

Below, a simplified version of the derivation above. The letter Σ stands for the subderivation that goes from the hypothesis $(A \wedge B) \to D$ to $A \wedge B$:

$$
\dfrac{\dfrac{\vdash_i ((A \wedge B) \to (A \wedge B) \to D)) \to ((A \wedge B) \to D) \qquad [A \wedge B]^1 \qquad \begin{array}{c}[(A \wedge B) \to D]^2 \\ \Sigma \\ A \wedge B\end{array}}{A \wedge B}\,1,2}{A}
$$

The derivation above permutes to:

$$
\dfrac{\vdash_i (A \to (A \to D)) \to (A \to D) \qquad [A]^1 \qquad \begin{array}{c} \dfrac{\dfrac{\dfrac{[A \wedge B]^2}{A} \quad [A \to D]^3}{D}}{[(A \wedge B) \to D]}\,2 \\ \Sigma \\ \dfrac{A \wedge B}{A} \end{array}}{A}\,1,3
$$

$[H_{I\wedge}]$ permutation:

Consider the following derivation Π:

$$
\dfrac{\dfrac{\begin{array}{c}\Pi_1 \\ ((C \to D) \to D)\end{array} \quad \begin{array}{c}[C]^1 \\ \Pi_2 \\ A\end{array} \quad \begin{array}{c}[D]^2 \\ \Pi_3 \\ A\end{array}}{A}\,1,2 \qquad \dfrac{\begin{array}{c}\Pi_4 \\ (E \to F) \to F\end{array} \quad \begin{array}{c}[E]^3 \\ \Pi_5 \\ B\end{array} \quad \begin{array}{c}[F]^4 \\ \Pi_6 \\ B\end{array}}{B}\,3,4}{A \wedge B}
$$

First apply the $[Hp]$ reduction procedure:

$$
\dfrac{\dfrac{\vdash_i (A \to (A \to D)) \to (A \to D) \quad [A]^5 \quad \begin{array}{c} \dfrac{\dfrac{\dfrac{\begin{array}{c}[C]^3 \\ \Pi_2 \\ A\end{array} \quad [A \to D]^1}{D}}{C \to D}\,3 \quad \begin{array}{c}\Pi_1 \\ (C \to D) \to D\end{array}}{[D]} \\ \Pi_3 \\ A \end{array}}{A}\,5,1 \qquad \dfrac{\vdash_i (B \to (B \to F)) \to (B \to F) \quad [B]^6 \quad \begin{array}{c} \dfrac{\dfrac{\dfrac{\begin{array}{c}[E]^4 \\ \Pi_5 \\ B\end{array} \quad [B \to F]^2}{F}}{E \to F}\,4 \quad \begin{array}{c}\Pi_4 \\ (E \to F) \to F\end{array}}{[F]} \\ \Pi_6 \\ B \end{array}}{B}\,6,2}{A \wedge B}
$$

Let's simplify the derivation above. The letter Σ will be used alongside the letter Π:

$$\dfrac{\dfrac{\vdash_i (A\to(A\to D))\to(A\to D)\qquad [A]^3\qquad \begin{array}{c}[A\to D]1\\ \Sigma_1\\ A\end{array}}{A}\,3,1 \qquad \dfrac{\vdash_i (B\to(B\to F))\to(B\to F)\qquad [B]^4\qquad \begin{array}{c}[B\to F]^2\\ \Sigma_2\\ B\end{array}}{B}\,4,2}{A\wedge B}$$

The derivation above permutes to:

$[H_{I\wedge}]$ permutation procedure:[95]:

$$\dfrac{\vdash_i ((A\wedge B)\to((A\wedge B)\to D))\to((A\wedge B)\to D)\qquad [A\wedge B]^5\qquad \dfrac{\vdash_i ((A\wedge B)\to((A\wedge B)\to F))\to((A\wedge B)\to F)\qquad [A\wedge B]^5\qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{[A]^3\qquad \begin{array}{c}\dfrac{\dfrac{\dfrac{[A]^3\quad [B]^4}{A\wedge B}\quad [(A\wedge B)\to F]^1}{F}}{B\to F}\,4\\ \Sigma_2\\ B\end{array}}{A\wedge B}\qquad [(A\wedge B)\to D]^2}{\dfrac{D}{A\to D}\,3}\\ \Sigma_1\\ A\end{array}\qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[A]^3\quad [B]^4}{A\wedge B}\quad [(A\wedge B)\to D]^2}{D}}{A\to D}\,3\\ \Sigma_1\\ A\end{array}\qquad [B]^4}{A\wedge B}\qquad [(A\wedge B)\to F]^1}{\dfrac{F}{B\to F}\,4}\\ \Sigma_2\\ B\end{array}}{A\wedge B}\,1,5}{A\wedge B}\,2,5}{A\wedge B}$$

Below, a simplified version where the applications of the H rule have been omitted:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{[A]^3\qquad \begin{array}{c}\dfrac{\dfrac{\dfrac{[A]^3\quad [B]^4}{A\wedge B}\quad [(A\wedge B)\to F]^1}{F}}{B\to F}\,4\\ \Sigma_2\\ B\end{array}}{A\wedge B}\qquad [(A\wedge B)\to D]^2}{\dfrac{D}{A\to D}\,3}\\ \Sigma_1\\ A\end{array}\qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{[A]^3\quad [B]^4}{A\wedge B}\quad [(A\wedge B)\to D]^2}{D}}{A\to D}\,3\\ \Sigma_1\\ A\end{array}\qquad [B]^4}{A\wedge B}\qquad [(A\wedge B)\to F]^1}{\dfrac{F}{B\to F}\,4}\\ \Sigma_2\\ B\end{array}}{\begin{array}{c}H-rule\,\dfrac{A\wedge B}{A\wedge B}\,1\\ H-rule\,\dfrac{}{A\wedge B}\,2\end{array}}$$

4.4.1.2 $\vee$ permutations

- $[H_{I\vee}]$ permutation

$$\dfrac{\dfrac{(C\to D)\to D\qquad \begin{array}{c}[C]^1\\ \Pi_1\\ A\end{array}\qquad \begin{array}{c}[D]^2\\ \Pi_2\\ A\end{array}}{A}\,1,2}{A\vee B}$$

Permutes to:

[95] this permutation procedure was developed by Professor Luiz Carlos Pereira

$$\dfrac{(C \to D) \to D \quad \dfrac{\begin{array}{c}[C]^1\\ \Pi_1\\ A\end{array}}{A \vee B} \quad \dfrac{\begin{array}{c}[D]^2\\ \Pi_2\\ A\end{array}}{A \vee B}}{A \vee B}\, 1,2$$

- $[HP_{I\vee}]$ permutation [96]

$$\dfrac{\dfrac{\vdash_i (A \to (A \to C)) \to (A \to C) \quad [A]^1 \quad \begin{array}{c}[A \to C]^2\\ \Pi\\ A\end{array}}{A}\, 1,2}{A \vee B}$$

Permutes to:

$$\dfrac{\vdash_i (A \to (A \to C)) \to (A \to C) \quad \dfrac{[A]^1}{A \vee B} \quad \dfrac{\begin{array}{c}[A \to C]^2\\ \Pi\\ A\end{array}}{A \vee B}}{A \vee B}\, 1,2$$

.

Definition 4.4.3 (**$H_{E\vee}$-permutation**)

- (a) Minor Premise

$$\dfrac{\begin{array}{c}\Pi_1\\ (A \vee B)\end{array} \quad \dfrac{\begin{array}{c}[A]^1\\ \Pi_2\\ ((E \to F) \to F)\end{array} \quad \begin{array}{c}[E]^3\\ \Pi_4\\ C\end{array} \quad \begin{array}{c}[F]^4\\ \Pi_5\\ C\end{array}}{C}\, 3,4 \quad \dfrac{\begin{array}{c}[B]^2\\ \Pi_3\\ ((G \to H) \to H)\end{array} \quad \begin{array}{c}[G]^5\\ \Pi_6\\ C\end{array} \quad \begin{array}{c}[H]^6\\ \Pi_7\\ C\end{array}}{C}\, 5,6}{C}\, 1,2$$

The derivation above permutes to:

$[H_{E\vee m}]$ permutation procedure

[96] This reduction procedure is a subcase of $[H_{I\vee}]$ where the $[H_{I\vee}]$ procedure is being applied after the $[HP]$ procedure.

$$\dfrac{\vdash_i (C\to(C\to H))\to(C\to H) \quad [C]^{c2} \quad \dfrac{\vdash_i (C\to(C\to F))\to(C\to F) \quad [C]^{c1} \quad \dfrac{\begin{array}{c}\Pi_1\\ A\vee B\end{array} \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}[E]^3\\ \Pi_4\\ C\end{array} \quad [C\to F]^4}{F}}{E\to F}\,3 \quad \begin{array}{c}[A]^1\\ \Pi_2\\ (E\to F)\to F\end{array}}{[F]}\\ \Pi_5\\ C\end{array} \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}[G]^5\\ \Pi_6\\ C\end{array} \quad [C\to H]^6}{H}}{G\to H}\,5 \quad \begin{array}{c}[B]^2\\ \Pi_2\\ (G\to H)\to H\end{array}}{[H]}\\ \Pi_7\\ C\end{array}}{C}\,1,2}{C}\,4,c1}{C}\,6,c2$$

- (b) Major premise

  Consider the following derivation, where the H rule is the major premise of $E\vee$:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ (E\to F)\to F\end{array} \quad \begin{array}{c}[E]^3\\ \Pi_2\\ A\vee B\end{array} \quad \begin{array}{c}[F]^4\\ \Pi_3\\ A\vee B\end{array}}{A\vee B}\,3,4 \quad \begin{array}{c}[A]^1\\ \Pi_4\\ C\end{array} \quad \begin{array}{c}[B]^2\\ \Pi_5\\ C\end{array}}{C}\,1,2$$

  First, apply the [HP] reduction procedure:

$$\dfrac{\dfrac{\vdash_i ((A\vee B)\to((A\vee B)\to F))\to(((A\vee B)\to F) \quad [A\vee B]^5 \quad \begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}[E]^3\\ \Pi_2\\ A\vee B\end{array} \quad [(A\vee B)\to F]^4}{[F]}}{E\to F}\,3 \quad \begin{array}{c}\Pi_1\\ (E\to F)\to F\end{array}}{[F]}\\ \Pi_3\\ A\vee B\end{array}}{A\vee B}\,4,5 \quad \begin{array}{c}[A]^1\\ \Pi_4\\ C\end{array} \quad \begin{array}{c}[B]^2\\ \Pi_5\\ C\end{array}}{C}\,1,2$$

  the derivation above permutes to [97]
  $[H_{E\vee M}]$ permutation procedure:

$$\dfrac{\vdash_i ((A\vee B)\to((A\vee B)\to F))\to(((A\vee B)\to F) \quad \dfrac{[A\vee B]^5 \quad \begin{array}{c}[A]^1\\ \Pi_4\\ C\end{array} \quad \begin{array}{c}[B]^2\\ \Pi_5\\ C\end{array}}{C}\,1,2 \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{c}[E]^3\\ \Pi_2\\ A\vee B\end{array} \quad [(A\vee B)\to F]^4}{[F]}}{E\to F}\,3 \quad \begin{array}{c}\Pi_1\\ (E\to F)\to F\end{array}}{[F]}\\ \Pi_3\\ A\vee B\end{array} \quad \begin{array}{c}[A]^1\\ \Pi_4\\ C\end{array} \quad \begin{array}{c}[B]^2\\ \Pi_5\\ C\end{array}}{C}\,1,2}{C}$$

[97] the procedure below is Prawitz's permutation procedure for $E\vee$ present in (PRAWITZ, 1965, pg.51):

$$\dfrac{\begin{array}{c}\Pi_1\\ (E\to F)\to F\end{array} \quad \dfrac{\begin{array}{c}[E]^3\\ \Pi_2\\ A\vee B\end{array} \quad \begin{array}{c}[A]^1\\ \Pi_4\\ C\end{array} \quad \begin{array}{c}[B]^2\\ \Pi_5\\ C\end{array}}{C}\,1,2 \quad \dfrac{\begin{array}{c}[F]^4\\ \Pi_3\\ A\vee B\end{array} \quad \begin{array}{c}[A]^1\\ \Pi_4\\ C\end{array} \quad \begin{array}{c}[B]^2\\ \Pi_5\\ C\end{array}}{C}\,1,2}{C}\,3,4$$

applied to an H rule that has already gone through the [HP] reduction.

4.4.2 Normalization proof

4.4.2.1 Definitions

Definition 4.4.4 *Critical derivation*

A derivation Π is called a critical derivation iff it satisfies the following conditions:

- *The last rule applied in Π is not an application of the H-Rule.*
- *The derivation(s) of the premiss(es) of the last rule applied in Π is (are) a $GH-derivation(s)$.*

Definition 4.4.5 *Intuitionistic part: Given a derivation Π, the intuitionistic parts of Π are a sequence of steps in Π that contains no applications of the $H-rule$*

Definition 4.4.6 *Classical part: Given a derivation Π, the classical parts of Π are sequences of steps of Π that contains only applications of the $H-rule$*

Definition 4.4.7 *General Hosoi Normal Form (GHNF)*

A GH-derivation is said to be in GH-normal-form (GHNF) iff its intuitionistic part is normal, i.e, no formula in the intuitionistic part is at the same time the conclusion of an I rule and major premisse of an E rule and there are no maximal segments.

4.4.2.2 Normalization

Lemma 4.4.8 (Critical Lemma) *Let Π be a critical derivation of A from Γ in NH. Then Π can be tranformed into a GH-derivation Π' of A from Γ.*

The proof is by induction on the number of applications of the H-rule in Π

Basis In case Π does not contain any application of the H-rule, then we take $\Pi^{'} = \Pi$

Inductive Step

- **1.** The last rule applied in Π is $E\wedge$ and Π has k applications of the H-rule. Π has the following form:

$$\dfrac{\vdash_i ((C\wedge D)\to(C\wedge D\to P_k))\to((C\wedge D)\to P_k)\qquad [C\wedge D]^{ck}\qquad \begin{array}{c}\dfrac{\vdash_i ((C\wedge D)\to(C\wedge D\to P_1))\to((C\wedge D)\to P_1)\qquad [C\wedge D]^{c1}\qquad \begin{array}{c}[(C\wedge B)\to P_1]^{p1}\quad \ldots\quad [(C\wedge B)\to P_k]^{pk}\\ \Pi_i\\ C\wedge D\end{array}}{C\wedge D}\,c1,p1\\ \vdots\\ C\wedge D\end{array}}{\dfrac{C\wedge B}{C}}\,ck,pk$$

Apply the $[HP_{\wedge_E}]$ procedure:

$$\dfrac{\vdash_i (C \to (C \to P_k)) \to (C \to P_k)) \qquad [C]^{ck} \qquad \dfrac{\dfrac{\vdash_i ((C \wedge D) \to (C \wedge D \to p_1)) \to ((C \wedge D) \to p_1)) \qquad [C \wedge D]^{ck-1} \qquad \begin{matrix} \dfrac{\vdash_i ((C \wedge D) \to (C \wedge D \to P_1)) \to ((C \wedge D) \to P_1)) \qquad [C \wedge D]^{c1} \qquad \begin{matrix} [(C \wedge B) \to P_1]^1 \quad \dots \quad [C \to P_{k-1}]^{k-1} \qquad \dfrac{\dfrac{\dfrac{[C \wedge D]^{c \wedge d}}{C} \quad [C \to P_k]^k}{P_k}}{[(C \wedge B) \to p_k]}\, c \wedge d \\ \Pi_i \\ C \wedge D \end{matrix}}{C \wedge D}\, p1, c1 \\ \vdots \\ C \wedge D \end{matrix}}{C \wedge B}\, ck-1, pk-1}{C}}{C}\, ck, pk$$

The result now follows directly from the induction hypothesis applied to the derivation of the premise of the last application of the H rule in the derivation above.

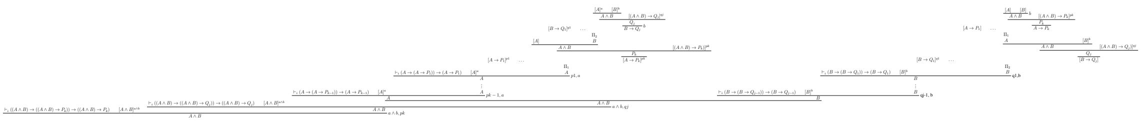
If we apply this procedure k times we get:

$$\dfrac{\vdash_i (C \to (C \to P_k)) \to (C \to P_k) \qquad [C]^{ck} \qquad \begin{matrix} \dfrac{\vdash_i ((C \to (C \to P_1)) \to ((C \to P_1)) \qquad [C]^{c1} \qquad \begin{matrix} \dfrac{\dfrac{\dfrac{[C \wedge D]^{c \wedge d}}{C} \quad [C \to P_1]^1}{P_1}}{[(C \wedge B) \to P_1]}\, c \wedge d \quad \dots \quad \dfrac{\dfrac{\dfrac{[C \wedge D]^{c \wedge d}}{C} \quad [C \to P_k]^k}{P_k}}{[(C \wedge B) \to P_k]}\, c \wedge d \\ \Pi_i \\ \dfrac{C \wedge D}{C} \end{matrix}}{C}\, c1, 1 \\ \vdots \\ C \end{matrix}}{C}\, ck, k$$

- **2.** The last rule applied in Π is $I\wedge$ and Π has k applications of the H-rule. Π has the following form:

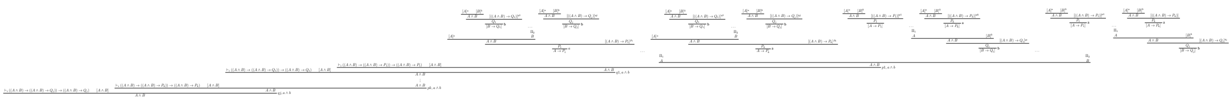
$$\dfrac{\dfrac{\vdash_i (A \to (A \to P_k)) \to (A \to P_k) \qquad [A]^a \qquad \begin{matrix} \dfrac{\vdash_i (A \to (A \to P_1)) \to (A \to P_1) \qquad [A]^a \qquad \begin{matrix} [A \to P_1]^{p1} \quad \dots \quad [A \to P_k]^{pk} \\ \Pi_1 \\ A \end{matrix}}{A}\, p1, a \\ \vdots \\ A \end{matrix}}{A}\, pk, a \qquad \dfrac{\vdash_i (B \to (B \to Q_j)) \to (B \to Q_j) \qquad [B]^b \qquad \begin{matrix} \dfrac{\vdash_i (B \to (B \to Q_1)) \to (B \to Q_1) \qquad [B]^b \qquad \begin{matrix} [B \to Q_1]^{q1} \quad \dots \quad [B \to Q_j]^{qj} \\ \Pi_2 \\ A \end{matrix}}{B}\, q1, b \\ \vdots \\ B \end{matrix}}{B}\, qj, b}{A \wedge B}$$

Apply the $[H_{I\wedge}]$ permutation procedure:



The result now follows directly from the induction hypothesis applied to the derivation of the premise of the last application of the H rule in the derivation above.

If we apply this procedure K times we get:



Below, a simplified version:

$$\dfrac{\begin{matrix} \dfrac{\dfrac{[A]^a \quad \begin{matrix} \Delta_1 \ \dots \ \Delta_j \\ \Pi_2 \\ B \end{matrix}}{A \wedge B} \quad [(A \wedge B) \to P_1]^{P_1}}{\dfrac{P_1}{[A \to P_1]}\, a} \quad \dots \quad \dfrac{\dfrac{[A]^a \quad \begin{matrix} \Delta_1 \ \dots \ \Delta_j \\ \Pi_2 \\ B \end{matrix}}{A \wedge B} \quad [(A \wedge B) \to P_k]^{P_k}}{\dfrac{P_k}{[A \to P_k]}\, \text{a}} \\ \Pi_1 \\ A \end{matrix} \qquad \begin{matrix} \dfrac{\dfrac{\begin{matrix} \Sigma_1 \ \dots \ \Sigma_k \\ \Pi_1 \\ A \end{matrix} \quad [B]^b}{A \wedge B} \quad [(A \wedge B) \to Q_1]^{q_1}}{\dfrac{Q_1}{[B \to Q_1]}\, \text{b}} \quad \dots \quad \dfrac{\dfrac{\begin{matrix} \Sigma_1 \ \dots \ \Sigma_k \\ \Pi_1 \\ A \end{matrix} \quad [B]^b}{A \wedge B} \quad [(A \wedge B) \to Q_j]^{q_j}}{\dfrac{Q_j}{[B \to Q_j]}\, \text{b}} \\ \Pi_2 \\ B \end{matrix}}{\begin{matrix} \text{H rule}\ \dfrac{A \wedge B}{\text{H rule}\ \dfrac{A \wedge B}{A \wedge B}\, q1, a \wedge b}\, p1, a \wedge b \\ \vdots \\ \text{H rule}\ \dfrac{A}{\text{H rule}\ \dfrac{A \wedge B}{A \wedge B}\, qj, a \wedge b}\, pk, a \wedge b \end{matrix}}$$

where the applications of the H rule have been omitted and

- Σ_i stands for:

$$\dfrac{\dfrac{\dfrac{[A]^a \qquad [B]^b}{A \wedge B} \qquad [((A \wedge B) \to P_i)]^{pi}}{P_i}}{[A \to P_i]}\, \text{a}$$

- Δ_i stands for:

$$\dfrac{\dfrac{\dfrac{[A]^a \qquad [B]^b}{A \wedge B} \qquad [((A \wedge B) \to Q_i)]^{qi}}{Q_1}}{[B \to Q_i]}\ \mathrm{b}$$

- **3.** The last rule applied in Π is $I\vee$ and Π has k applications of the H-rule. Π has the following form:

$$\dfrac{\dfrac{\vdash_i (C \to (C \to P_k)) \to (C \to P_k)) \qquad [C]^{ck} \qquad \begin{matrix} \dfrac{\vdash_i ((C \to (C \to P_1)) \to ((C \to P_1)) \qquad [C]^{c1} \qquad \begin{matrix} [(C \to P_1]^{p1} \quad \dots \quad [(C \to P_k]^{pk} \\ \Pi_i \\ C \end{matrix}}{C}\ c1, p1 \\ \vdots \\ C \end{matrix}}{C}\ ck, pk}{C \vee B}$$

Apply the $[H_{I\vee}]$ reduction procedure

$$\dfrac{\vdash_i (C \to (C \to P_k)) \to (C \to P_k)) \qquad \dfrac{[C]^{ck}}{C \vee D} \qquad \dfrac{\dfrac{\vdash_i (C \to (C \to P_{k-1})) \to (C \to P_{k-1})) \qquad [C]^{ck-1} \qquad \begin{matrix} \dfrac{\vdash_i (C \to (C \to P_1)) \to (C \to P_1) \qquad [C]^{c1} \qquad \begin{matrix} [(C \to P_1]^{p1} \quad \dots \quad [(C \to P_k]^{pk} \\ \Pi_i \\ C \end{matrix}}{C}\ c1, p1 \\ \vdots \\ C \end{matrix}}{C}\ ck-1, pk-1}{C \vee B}}{C \vee D}\ ck, pk$$

The result now follows directly from the induction hypothesis applied to the derivation of the premise of the last application of the H rule in the derivation above.

If we apply this procedure k times we get:

$$\dfrac{\vdash_i (C \to (C \to P_k)) \to (C \to P_k) \qquad \dfrac{[C]^{ck}}{C \vee D} \qquad \begin{matrix} \dfrac{\vdash_i (C \to (C \to P_1)) \to (C \to P_1) \qquad \dfrac{[C]^{c1}}{C \vee D} \qquad \begin{matrix} [(C \to P_1]^{p1} \quad \dots \quad [(C \to P_k]^{pk} \\ \Pi_i \\ \dfrac{C}{C \vee D} \end{matrix}}{C \vee D}\ c1, p1 \\ \vdots \\ C \vee D \end{matrix}}{C \vee D}\ ck, pk$$

- **4.** The last rule applied in Π is $E\vee$ and Π has k applications of the H-rule. The major premise of the $E\vee$ rule is the conclusion of an H rule

(a) Π has the following form:

$$\dfrac{\dfrac{\vdash_i ((A \vee B) \to ((A \vee B) \to P_k)) \to ((A \vee B) \to P_k) \qquad [(A \vee B)]^{a \vee b} \qquad \begin{matrix} \dfrac{\vdash_i ((A \vee B) \to ((A \vee B) \to P_1)) \to ((A \vee B) \to P_1) \qquad [(A \vee B)]^{a \vee b} \qquad \begin{matrix} [((A \vee B) \to P_1]^1 \quad \dots \quad [((A \vee B) \to P_k]^k \\ \Pi_1 \\ (A \vee B) \end{matrix}}{(A \vee B)}\ 1, a \vee b \\ \vdots \\ (A \vee B) \end{matrix}}{(A \vee B)}\ k, a \vee b \qquad \begin{matrix} [A]^a \\ \Pi_2 \\ C \end{matrix} \qquad \begin{matrix} [B]^b \\ \Pi_3 \\ C \end{matrix}}{C}\ a, b$$

If we apply the $[H_{E\vee M}]$ reduction procedure we get:

$$\dfrac{\vdash_i ((A \to B) \to ((A \to B) \to P_k)) \to ((A \to B) \to P_k) \qquad \dfrac{[A \vee B]^{a \vee b} \qquad \begin{matrix} [A]^a \\ \Pi_2 \\ C \end{matrix} \qquad \begin{matrix} [B]^b \\ \Pi_3 \\ C \end{matrix}}{C}\ a, b \qquad \dfrac{\dfrac{\vdash_i ((A \vee B) \to ((A \vee B) \to P_k - 1)) \to ((A \vee B) \to P_k - 1) \qquad [(A \vee B)]^{a \vee b} \qquad \begin{matrix} \dfrac{\vdash_i ((A \vee B) \to ((A \vee B) \to P_1)) \to ((A \vee B) \to P_1) \qquad [(A \vee B)]^{a \vee b} \qquad \begin{matrix} [((A \vee B) \to P_1]^1 \quad \dots \quad [((A \vee B) \to P_k]^k \\ \Pi_1 \\ (A \vee B) \end{matrix}}{(A \vee B)}\ 1, a \vee b \\ \vdots \\ (A \vee B) \end{matrix}}{(A \vee B)}\ k-1, a \vee b \qquad \begin{matrix} [A]^a \\ \Pi_2 \\ C \end{matrix} \qquad \begin{matrix} [B]^b \\ \Pi_3 \\ C \end{matrix}}{C}\ a, b}{C}\ k, a \vee b$$

The result now follows directly from the induction hypothesis applied to the derivation of the premise of the last application of the H rule in the derivation above.

If we apply this procedure k times we get:

$$\dfrac{\vdash_i ((A\vee B)\to((A\vee B)\to P_k))\to((A\vee B)\to P_k) \quad \dfrac{[A\vee B]^{a\vee b} \quad \begin{array}{c}[A]^a\\ \Pi_2\\ C\end{array} \quad \begin{array}{c}[B]^b\\ \Pi_3\\ C\end{array}}{C}\,a,b \quad \begin{array}{c} \dfrac{\vdash_i ((A\vee B)\to((A\vee B)\to P_1))\to((A\vee B)\to P_1) \quad \dfrac{[A\vee B]^{a\vee b} \quad \begin{array}{c}[A]^a\\ \Pi_2\\ C\end{array} \quad \begin{array}{c}[B]^b\\ \Pi_3\\ C\end{array}}{C}\,a,b \quad \dfrac{\begin{array}{c}[(A\vee B)\to P_1]^1 \quad \dots \quad [((A\vee B)\to P_k]^k\\ \Pi_1\\ A\vee B\end{array} \quad \begin{array}{c}[A]^a\\ \Pi_2\\ C\end{array} \quad \begin{array}{c}[B]^b\\ \Pi_3\\ C\end{array}}{C}\,a,b}{C}\,1,a\vee b \\ \vdots \\ C \end{array}}{C}\,k,a\vee b$$

- **5.** The last rule applied in Π is $E\vee$ and Π has $k+j$ applications of the H-rule. The minor premise(s) of the $E\vee$ is the (are) conclusion(s) of an H rule.

  Π has the following form:

$$\dfrac{\begin{array}{c}\Pi_1\\ A\vee B\end{array} \quad \dfrac{\vdash_i (C\to(C\to P_k))\to(C\to P_k) \quad [C]^{ck} \quad \begin{array}{c} \dfrac{\vdash_i (C\to(C\to P_1))\to(C\to P_1) \quad [C]^{c1} \quad \begin{array}{c}[A]^1 \quad [C\to P_1]^{p1} \quad \dots \quad [C\to P_k]^{pk}\\ \Pi_2\\ C\end{array}}{C}\,c1,p1 \\ \vdots \\ C\end{array}}{C}\,ck,pk \quad \dfrac{\vdash_i (C\to(C\to Q_j))\to(C\to Q_j) \quad [C]^{cj} \quad \begin{array}{c} \dfrac{\vdash_i (C\to(C\to Q_1))\to(C\to Q_1) \quad [C]^{c1} \quad \begin{array}{c}[B]^2 \quad [C\to Q_1]^{q1} \quad \dots \quad [C\to Q_j]^{qj}\\ \Pi_3\\ C\end{array}}{C}\,c1,q1 \\ \vdots \\ C\end{array}}{C}\,qj,cj}{C}\,1,2$$

  If we apply the $[H_{E\vee m}]$ reduction procedure we get:

$$\dfrac{\vdash_i (C\to(C\to Q_j))\to(C\to Q_j) \quad [C]^{cj} \quad \dfrac{\vdash_i (C\to(C\to P_k))\to(C\to P_k) \quad [C]^{ck} \quad \dfrac{\begin{array}{c}\Pi_1\\ A\vee B\end{array} \quad \dfrac{\vdash_i (C\to(C\to P_{k-1}))\to(C\to P_{k-1}) \quad [C]^{ck-1} \quad \begin{array}{c}\dfrac{\vdash_i (C\to(C\to P_1))\to(C\to P_1) \quad [C]^{c1} \quad \begin{array}{c}[C\to P_1]^{p1} \quad \dots \quad [C\to P_k]^{pk} \quad [A]^1\\ \Pi_2\\ C\end{array}}{C}\,c1,q1 \\ \vdots \\ C\end{array}}{C}\,pk-1,ck-1 \quad \dfrac{\vdash_i (C\to(C\to Q_{j-1}))\to(C\to Q_{j-1}) \quad [C]^{cj-1} \quad \begin{array}{c}\dfrac{\vdash_i (C\to(C\to Q_1))\to(C\to Q_1) \quad [C]^{c1} \quad \begin{array}{c}[C\to Q_1]^{q1} \quad \dots \quad [C\to Q_j]^{qj} \quad [B]^2\\ \Pi_3\\ C\end{array}}{C}\,c1,q1 \\ \vdots \\ C\end{array}}{C}\,qj-1,cj-1}{C}\,1,2}{C}\,pk,ck}{C}\,qj,cj$$

The result now follows directly from the induction hypothesis applied to the derivation of the premise of the last application of the H rule in the derivation above:

$$\dfrac{\vdash_i (C\to(C\to P_k))\to(C\to P_k) \quad [C]^{ck} \quad \dfrac{\vdash_i (C\to(C\to P_1))\to(C\to P_1) \quad [C]^{c1} \quad \begin{array}{c}\dfrac{\vdash_i (C\to(C\to Q_j))\to(C\to Q_j) \quad [C]^{cj} \quad \begin{array}{c}\dfrac{\vdash_i (C\to(C\to Q_1))\to(C\to Q_1) \quad [C]^{c1} \quad \begin{array}{c}[C\to P_1]^{p1} \quad \dots \quad [C\to P_k]^{pk} \quad [A]^1 \quad [C\to Q_1]^{q1} \quad \dots \quad [C\to Q_j]^{qj} \quad [B]^2 \\ \dfrac{\begin{array}{c}\Pi_1\\ A\vee B\end{array} \quad \begin{array}{c}\Pi_2\\ C\end{array} \quad \begin{array}{c}\Pi_3\\ C\end{array}}{C}\,1,2\end{array}}{C}\,c1,q1 \\ \vdots \\ C\end{array}}{C}\,qj,cj \\ \vdots \\ C\end{array}}{C}\,p1,c1}{C}\,pk,ck$$

Theorem 4.4.9 *Theorem (Normalization). If Π is a derivation of A from Γ in NH, then Π reduces to a GHNF derivation Π^* of A from Γ.*

The proof proceeds by induction on the length of Π.

Basis: The base case is trivial.

Inductive step

1. The last rule applied in Π is not an application of the H-rule. By the induction hypothesis, the derivations of each premise of Π can be reduced to GH-derivations in normal form. By the critical lemma, we obtain a GH-derivation Π' of A from Γ. A normal derivation Π^* of A from Γ can be obtained by the usual normalization technique applied to the intuitionistic part of Π'.

2. The last rule applied in Π is an application of a H-rule. The result follows directly from the induction hypothesis.

Corollary 4.4.10 *Logical Separability: For a provable formula in the NH system, intuitionistic reasoning can be separated from classical reasoning.*

Proof: this follows directly from the fact that every derivation Π of A from Γ can be transformed into a normal derivation Π^* in such a way that Π^* is divided into an intuitionistic part and a classical part.

4.4.3 Separability proof

Hosoi's proof of separability relies on a transformation procedure from LK to his system. Our proof will be a bit different. It depends on the normalization procedure result obtained in this section. Our strategy relies on the modular nature of Natural Deduction systems, and on the fact that we can divide our proofs into an intuitionistic part and a classical part.

Theorem 4.4.11 *[Hosoi's] Separation Theorem: For a provable formula in the system, there is a proof in which are used only the rules for implication and the rules for the other logical symbols actually appearing in the formula.*

Proof: First, take any derivation Π in the NH system. If Π is intuitionistic and if Π is a normal derivation, it is well known that it is separable. As it was shown, if Π is a classical derivation in NH, Π can be reduced to a GHNF Π^* derivation. Π^* is divided into two parts, an intuitionistic part and a classical part (that consists of a pile of applications of the $H-rule$). Since the Intuitionistic part of Π^* is separable, and since the Classical part is a series of applications of the $H - rule$ (an implicational rule), it follows that for a provable formula in the system, there is a proof in which are used only rules for implication and rules for the other logical symbols actually appearing in the formula. NH is separable in Hosoi's sense.

□

Hosoi's separation theorem can be strengthened. In his formulation, implication has a special place. In the case of NH, we can actually show that every provable formula contains an implication. This follows from the fact that (i) negation is defined as $A \rightarrow \bot$, which means that every formula containing negation is a formula that contains implication, and (ii) the $< \wedge, \vee >$ fragment of NH does not contain any theorem [98]. From these two facts, it follows that every theorem of NH has an implication. It then follows that we need not to mention implication in the formulation of the separation theorem for NH. The theorem can thus be stated as follows:

Theorem 4.4.12 *[Stronger] Separation Theorem: For a provable formula in the NH system, there is a proof in which are used only the rules for the logical symbols actually appearing in the formula.*

[98] one way to explain this is through the fact that the introduction rules for $\wedge$ and $\vee$ cannot discharge assumptions. There is no way to close derivations!

CONCLUSIONS

Although Leblanc's theorem affirms that "If either Double Negation Elimination or classical reductio (or some equivalent rule) are taken to partly determine the meaning of classical negation, then no complete natural deduction formalization of classical logic is separable" (LEBLANC, 1966, pg.35), it is not true that there can be no separable formalization of classical logic in natural deduction. In fact, as we have seen, there are formalizations that even have the subformula principle.

This leads us to the discussion about the analyticity of these systems. Murzi's classification of Strictly and Ultra strictly Analytical systems is very useful in this context. Classical logic can be formulated in a Natural Deduction context in ways that (i) dont respect neither the subformula nor the separation principle, (ii) in ways that only respect the separation principle and (iii) in ways that respect both the subformula and the separation principle:

Classical System	Subformula	Separability
NJ + DNE		
NJ + Reductio		
NH		X
NCP+	X	X
NJ + Restart	X	X

The NH system is interesting because it explicitates the difference between the subformula principle and the separation principle. Because separability is a[99] corollary of the subformula principle, the two notions could be mistakenly treated as equivalent. But, NH doesn't have the subformula property. Although Subformula and Separability are properties that indicate an analytical character of systems, they speak of different kinds of analyticity, i.e different 'modes' of containment.

The subformula property guarantees that the information used in every inferential step of the proof is contained in the final conclusion of the proof: if a formula A is provable in an Ultra Strictly Analytical system Θ, only subformulas of A will be used in its proof. Another way of reading the theorem is: if a formula A is provable at all in an Ultra Strictly Analytical system Θ, then A can be proved by 'combining' the parts of A. It is in this sense that we can say that in a Ultra Strict Analytical system every information that is needed to prove a formula A is already contained in A

The Separation principle is weaker. It guarantees that, given a provable formula A,

[99] see (PRAWITZ, 1971, pg.54)

every 'type of reasoning' [100] used in the proof of A is already indicated by the logical operators present in A. If, in a Strictly Analytical System, a formula A has implication and conjunction for example, the separation property guarantees that there is a proof of A in which only the rules for implication and conjunction will be used in order to prove A. It is in this sense that one could say that in a Strictly Analytical System the types of reasoning used in order to prove a formula A are already contained in A.

NH is also interesting because it is the only system in the table above that gives NJ the foundational status [101]. NPC+ and NJ+restart aren't enough to give NJ the foundational status because they are obtained from NJ through structural changes. It is the same reason with LK and LJ. NH is a *Strictly Analytic System*. It is not known to the autor if there exists a *Ultra Strictly Analytic System* constructed from a foundational system through the addition of an operational rule. For future work, (1) it is still to be investigated if it is possible to construct a *Ultra Strictly Analytic System* in natural deduction by adding an operational rule, i.e, if it is possible to construct a classical natural deduction system that (i) respects the subformula principle and (ii) is obtained through the addition of an operational rule to NJ.

The normalization technique used for NH also enables us to talk about another sense of separation: that in every derivation Π, intuitionistic reasoning can be separated from classical reasoning. This sense of separation concerns not logical operators, but different types of logic.

The main results of this dissertation were: (1) the development of a strictly analytical system for propositional classical logic [NH] consisting of the propositional fragment of NJ plus Hosoi's rule (2) the development of a normalization procedure for NH and (3) the classification of different systems into strictly analytical and ultra strictly analytical. The strictly analytical system explored throughout this work was obtained by the addition of a classicizing implicational rule to the propositional fragment of NJ. Both ultra strictly analytical systems were obtained through the addition of structural rules to an intuitionistic system. The NH system still needs to be extended to first order logic.

[100] By types of reasoning one should understand the usage of different types of operational rules. The implicational rules could be characterized as the 'implicational type' of reasoning, the conjunction rules as the 'conjunctive type' of reasoning and so on.

[101] Although NP could also serve this purpose

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